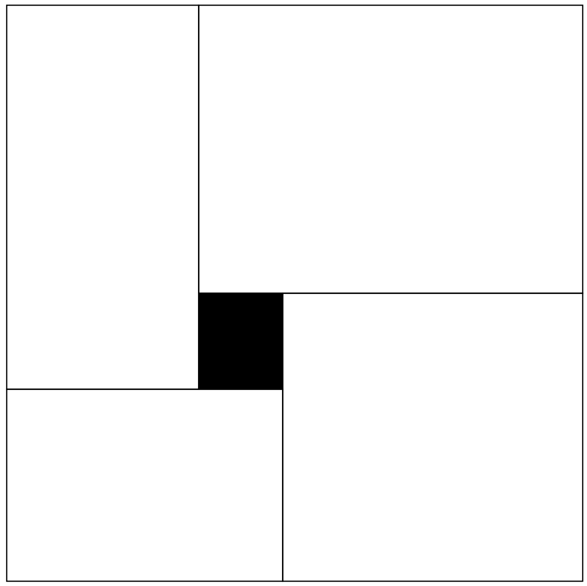
Solution Set 8

Problem 1 *Consider a $(2n+1)\times(2n+1)$ checker board, with the corners colored black. Suppose that we remove any black square from the board, living $4n^2+4n$ squares behind. Show that the remaining squares can be tiled with dominos. (A direct proof is probably easier than appealing to Hall's marriage theorem.)*

Surround the hole with 4 rectangles, as shown:



There are two case: The hole is an even distance from every edge, or an odd distance from every edge. Either way, each of these rectangles is even $\times$ odd, and thus can be tiled with dominos.

Problem 2 *Let G be a graph. By definition, a **perfect matching** of G is a collection M of edges of G so that every vertex of G lies on exactly one edge of M. Let $(v_1, v_2, \ldots, v_{2k})$ be a cycle of G of even length. If M is a perfect matching of G which contains the edges (v_1, v_2), (v_3, v_4), $\ldots$, (v_{2k-1}, v_{2k}), then define the **twist** of M along $(v_1, v_2, \ldots, v_{2k})$ to be the perfect matching which deletes the edges (v_1, v_2), (v_3, v_4), $\ldots$, (v_{2k-1}, v_{2k}) from M and replaces them by the edges (v_2, v_3), (v_4, v_5), $\ldots$, (v_{2k-2}, v_{2k-1}), (v_{2k}, v_1).*

(a) If M and M' are two perfect matchings of G, show that we can change M to M' by a sequence of twists along various cycles of G.

Superimposing the matchings M and M' gives a 2-regular graph, which must be a union of cycles and doubled edges. Twisting along the cycles changes M to M'.

*An **induced cycle** of G is a cycle $(v_1, v_2, \ldots, v_m)$ so that G has no edges among the vertices v_i other than the m edges of the cycle.*

(b) Prove or disprove: If M and M' are two perfect matchings of G, then we can change M to M' by a sequence of twists along various induced cycles of G.

This is false. Consider the graph K_4, which has no induced cycles of even length at all. So there are no induced cycles to twist along, yet K_4 has three matchings.

(c) Prove or disprove: If G is bipartite and M and M' are two perfect matchings of G, then we can change M to M' by a sequence of twists along various induced cycles of G.

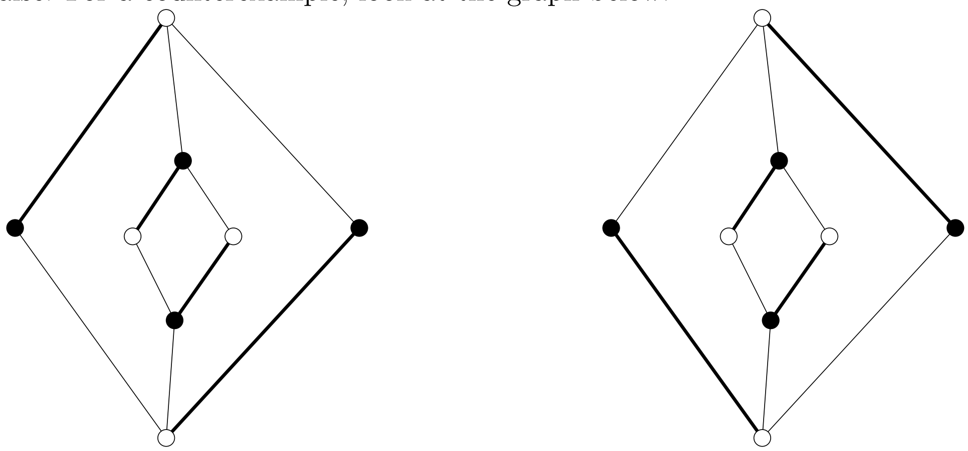
This is true. Let $M \to M_1 \to M_2 \to \cdots \to M'$ be a sequence of twists taking M to M' so that the lengths of the cycles used for twisting are as short as possible. Suppose for the sake of contradiction that one of these cycles is not induced; call it $(v_1, \ldots, v_{2k})$, and that this is used in twisting M_r to M_{r+1}. The assumption that it is not induced means that there is some additional edge from v_i to v_j, other than the edges of the cycle. The fact that the graph is bipartite means that $i \not\equiv j \bmod 2$.

Without loss of generality, let $1 \le i < j \le 2k$. Then twisting along $(v_1, \ldots, v_{2k})$ is the concatenation of twisting along $(v_i, v_{i+1}, \ldots, v_{j-1}, v_j)$ and twisting along $(v_j, v_{j+1}, \ldots, v_{2k}, v_1, \ldots, v_{i-1}, v_i)$.

Let G be a graph drawn in the plane without crossing itself. We'll say that a cycle of G bounds a face if there are no edges of G inside the part of $\mathbb{R}^2$ enclosed by that cycle.

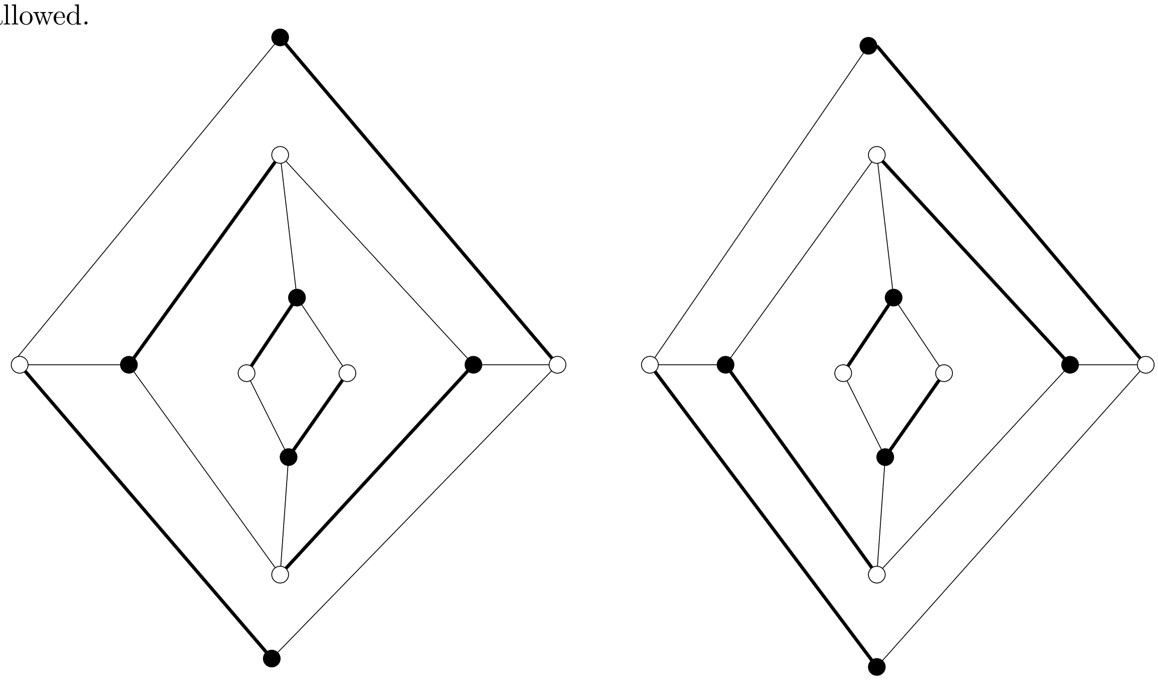
(d) (Harder) Prove or disprove: Let G be a connected bipartite planar graph and let M and M' be two perfect matchings. Then we can change M to M' by a sequence of twists along cycles which bound faces.

This is false. For a counterexample, look at the graph below.



There are 4 matchings of this graph, and twists join them into two pairs of two; the figure shows two matchings which cannot be joined.

The reader who is familiar with conventions for planar graphs may notice that these matchings can be joined if we allow twisting around the exterior face. But this doesn't salvage the statement. Here are two matchings which cannot be joined by twists around faces, even when the exterior face is allowed.



Problem 3 *We deal a deck of cards into 13 piles of 4. Show that it is possible to pick up one card from each pile and have precisely one card of each rank: One ace, one deuce and so forth, up to one king.*

Define a bipartite graph G whose "black" vertices are the piles and whose "white" vertices are the ranks of the cards, and where there is an edge from a black vertex to a white vertex if that rank appears in that pile. So this is a 4-regular bipartite graph with 13 vertices of each kind, and we want to show that it has a perfect matching.

Suppose otherwise. By Hall's theorem, there is a set S of piles such that fewer than $|S|$ different ranks appear in S. But the set S of piles contains $4|S|$ cards, and each rank can occur at most 4 times, so at least $|S|$ different ranks must occur in these piles; a contradiction.

Problem 4 *Let A be a square $n \times n$ matrix. I'll say that N is a **magic square** if there is some constant N such that every row and every column[1] of A sums to N. A **permutation matrix** is a $(0,1)$ matrix where every row and every column contains exactly one 1; so a permutation matrix is a magic square with $N = 1$.—*

(a) Show that, if A is a magic square with nonnegative integer entries, then A is a sum of N permutation matrices. (Hint in ROT13: Svefg gel gb fubj gung gurer vf n fvatyr crezhgngvba zngevk P fhpu gung $A - P$ fgvyy unf abaartngvir vagrtre ragevrf.)

We work by induction on N. The base case $N = 0$ is obvious.

Assume $N \geq 1$. Define a bipartite graph G with $2n$ vertices: One for each row and each column of A. Let there be an edge from the vertex for row i to the vertex for column j if $A_{ij} > 0$. We claim that G has a perfect matching.

Suppose otherwise. Then, by Hall's Marriage Theorem, there is some set R of rows and a smaller set C of columns so that the vertices indexed by R only border the columns indexed by C. Let's consider the sum of the A_{ij} for $i \in R$ and $j \in C$. Since $A_{ik} = 0$ when $i \in R$ and $k \notin C$, we have $\sum_{i \in R} \sum_{j \in C} A_{ij} = N|R|$. But, summing the columns first, we have $\sum_{j \in C} \sum_{i \in R} A_{ij} \leq N|C|$. So $N|R| \leq N|C|$, contradicting that C is supposed to be smaller than A.

So G has a matching, and this matching corresponds to a permutaion matrix P such that the entries of $A - P$ are nonnegative integers. We now apply the inductive hypothesis to write $A - P$ as a sum of permutation matrices.

(b) Show that, if A is a magic square with nonnegative real entries, then A is a positive linear combination of permutation matrices.

(c) (Harder) Let A be an $n \times n$ magic square with nonnegative real entries. Show that there are $(n-1)^2+1$ permutation matrices, $P_1, P_2, \ldots, P_{(n-1)^2+1}$ so that $A = \sum c_i P_i$ with the c_i nonnegative real numbers.

I'll first prove the easier bound of $n^2 - n + 1$. As before, we can find a permutation matrix P such that $A - cP$ has nonnegative entries for sufficiently small positive c. By choosing c correctly, we can make sure that $A - cP$ has one fewer positive entry than A does. Repeat this to subtract off $(n-1)^2 + 1$ permutation matrices; call the remainder R. So R is an $n \times n$ magic square, with $(n-1)n + 1$ entries which are zero. In particular, there must be some row of R which is entirely 0. But then the magic sum N must be zero, so R is zero, and we have written our original matrix as a sum of $(n-1)n + 1$ permutation matrices.

In fact, the correct bound is $(n-1)^2+1$. My original intended proof of this was broken, but here is a correct one. Let V be the vector space of all $n \times n$ matrices whose column and row sums are all equal. It is easy to check that this has dimension $(n-1)^2 + 1$. The result follows immediately from a result known as Cartheodroy's lemma: Let $v_1, v_2, \ldots, v_N$ be vectors in a vector space V of dimension d over $\mathbb{R}$. Let w be another vector in V, which is in the positive linear span of the v_i. Then w is in the positive linear span of some subset of the v_i with at most d elements.

Proof: Let e be the smallest integer such that w can be written as $\sum_{k=1}^{e} c_k v_{i_k}$ with the $c_k > 0$. We want to show that $e \leq d$. Assume otherwise. Then $v_{i_1}, v_{i_2}, \ldots, v_{i_e}$ are linearly dependent; say $\sum b_k v_{i_k} = 0$. Let j be the index for which $c_j/|b_j|$ is minimized. (If $b_j = 0$, we consider this ratio to be ∞.) Let α be the minimal value of $c_j/|b_j|$ and assume without loss of generality that $b_j > 0$.

[1] We don't impose this condition on the diagonals.

Then

$$\sum_{k=1}^{e}(c_k - \alpha b_k)v_{i_k} = \sum c_k v_{i_k} = w.$$

This has fewer nonzero terms than $\sum c_k v_{i_k}$, and all its coefficients are nonnegative, completing the contradiction.