Solution Set 9

Problem 1.(a) *Let G be a directed graph with source s, sink t and edge capacities $c(e)$. Suppose that all the capacities $c(e)$ are integers. Show that there is an optimal flow through G with integer amounts of flow through every edge.*

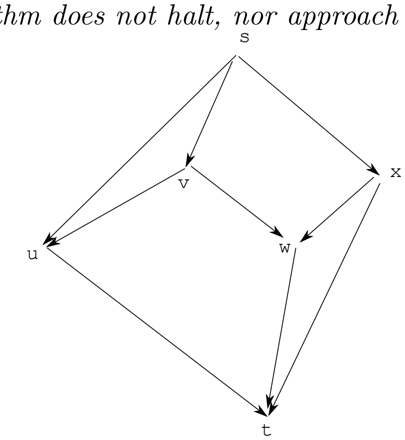
At every step of the Ford-Fulkerson algorithm, we increase or decrease the flow amount in each edge by an integer. When the algorithm terminates, we will have an optimal integer flow.

1.(b) *Let H be a bipartite graph. Construct a directed graph G, source s, sink t and some edge capacities $c(e)$, such that the maximal flow from s to t is equal to the maximum cardinality of any matching of G.*

Let the black vertices of H be $b_1, b_2, \ldots, b_p$ and the white vertices be $w_1, w_2, \ldots, w_q$. The vertices of G will be $s, b_1, b_2, \ldots, b_p, w_1, w_2, \ldots, w_q, t$. For each edge (b_i, w_j) of H, we will have an edge $b_i \longrightarrow w_j$ of G with capacity 1. There will also be edges $s \longrightarrow b_i$ of capacity 1 and $w_j \longrightarrow t$ of capacity 1 for all b_i and w_j.

By part (a), there is an optimal integral flow. In that flow, every edge will have flow 0 or 1. Let M be the set of edges of H which have flow 1. For any $b \in B$, the edge $s \longrightarrow b$ has capacity 1, so the outflow from b is at most 1 and at most 1 edge of M touches b. Similarly, for all $w \in W$, at most one edge of M touches w. So M is a matching. This shows that the optimal flow is $\leq$ the cardinality of the maximal matching, and this construction is clearly reversible to show that the maximal matching is achieved by a flow.

Problem 2 *The aim of this problem is to show an example of a graph with irrational edge weights where the Ford-Fulkerson algorithm does not halt, nor approach the correct limit.*



Our graph is shown in the image above. The capacities are as follows:

$$c(v,u) = c(v,w) = 1, \quad c(x,w) = \frac{\sqrt{5}-1}{2} \approx 0.618, \quad \text{all other edges have } c = 10.$$

2.(a) *Compute the maximum possible flow through G. Give an example of a cut whose capacity equals this flow.*

The optimal flow is 21. It can be achieved by putting 10 on $s \longrightarrow u \longrightarrow t$, 10 on $s \longrightarrow x \longrightarrow t$ and 1 on $s \longrightarrow v \longrightarrow w \longrightarrow t$. A corresponding cut is achieves this is $(\{s,u,v\},\{w,x,t\})$.

Define the paths $p_1 = (s \to x \to w \to v \to u \to t)$, $p_2 = (s \to v \to w \to x \to t)$ and $p_3 = (s \to u \to v \to w \to t)$.

2.(b) *Start with the flow which is 1 on $s \to v \to w \to t$. Successively increment it, as much as possible, along p_1, p_2, p_1, p_3, $\ldots$, with the patten repeating with period 4. Compute that first 4 flows you produce in this way and their residual graphs.*

The first several flows, and the sizes of the increases, are listed below:

$s \to u$	$s \to v$	$s \to x$	$v \to u$	$v \to w$	$x \to w$	$u \to t$	$w \to t$	$x \to t$	amount of increase
0	1	0	0	1	0	0	1	0	
0	1	τ	τ	$1-\tau$	τ	τ	1	0	τ
0	$1+\tau$	τ	τ	1	0	τ	1	τ	τ
0	$1+\tau$	1	1	τ	τ^2	1	1	τ	τ^2
τ^2	$1+\tau$	1	τ	1	τ^2	1	$1+\tau^2$	τ	τ^2

Some more experimentation suggests that the increases will continue τ^3, τ^3, τ^4, τ^4, τ^5, τ^5, τ^6, τ^6, ... The following table shows what the corresponding flows will be. We will check later that this is what happens:

path	increase	$s \to u$	$s \to v$	$s \to x$	$v \to u$	$v \to w$	$x \to w$	$u \to t$	$w \to t$	$x \to t$
p_1	τ^{2k-1}	0	0	τ^{2k-1}	τ^{2k-1}	$-\tau^{2k-1}$	τ^{2k-1}	τ^{2k-1}	0	0
p_2	τ^{2k-1}	0	τ^{2k-1}	0	0	$+\tau^{2k-1}$	$-\tau^{2k-1}$	0	0	τ^{2k-1}
p_1	τ^{2k}	0	0	τ^{2k}	τ^{2k}	$-\tau^{2k}$	τ^{2k}	τ^{2k}	0	0
p_3	τ^{2k}	τ^{2k}	0	0	$-\tau^{2k}$	τ^{2k}	0	0	τ^{2k}	0
	net increase				τ^{2k-1}	0	τ^{2k}			

We have left out the totals for the other columns, because they will be less important.

The total flows along the center edges at step m will be

m	$v \to u$	$v \to w$	$x \to w$
$4k$	$\tau + \tau^3 + \cdots + \tau^{2k-1}$	1	$\tau^2 + \tau^4 + \cdots + \tau^{2k}$
$4k+1$	$\tau + \tau^3 + \cdots + \tau^{2k-1} + \tau^{2k+1}$	$1 - \tau^{2k+1}$	$\tau^2 + \tau^4 + \cdots + \tau^{2k} + \tau^{2k+1}$
$4k+2$	$\tau + \tau^3 + \cdots + \tau^{2k-1} + \tau^{2k+1}$	1	$\tau^2 + \tau^4 + \cdots + \tau^{2k}$
$4k+3$	$\tau + \tau^3 + \cdots + \tau^{2k-1} + \tau^{2k+1} + \tau^{2k+2}$	$1 - \tau^{2k+2}$	$\tau^2 + \tau^4 + \cdots + \tau^{2k} + \tau^{2k+2}$

To evaluate these geometric series, note that $\tau + \tau^3 + \cdots + \tau^{2k-1} = (\tau - \tau^{2k})/(1-\tau^2) = (\tau - \tau^{2k+1})/\tau = 1 - \tau^{2k}$. So these sums are

	$v \to u$	$v \to w$	$x \to w$
$4k$	$1 - \tau^{2k}$	1	$\tau - \tau^{2k+1}$
$4k+1$	$1 - \tau^{2k+2}$	$1 - \tau^{2k+1}$	τ
$4k+2$	$1 - \tau^{2k+2}$	1	$\tau - \tau^{2k+1}$
$4k+3$	1	$1 - \tau^{2k+2}$	$\tau - \tau^{2k+3}$

OK, that computation was hypothetical: if the amount of increase cycled τ^{2k-1}, τ^{2k-1}, τ^{2k}, τ^{2k} ..., then these would be the resulting flows. To see that these are the amounts of the increases, we have to see that, at every step, some edge does come up to capacity (so we can't increase by more) and no edge goes beyond capacity (so we can increase by that much). From the last table, this is clear.

2.(c) *Show that, no matter how many times you go through the procedure in 2.(b), you'll never get to even half the total capacity of the network.*

The middle edges do approach their capacity. However, the flow along $s \to u$ increases by τ^{2k} in each cycle, so it never gets any larger than $\tau^2 + \tau^4 + \tau^6 + \cdots = \tau^2/(1-\tau^2) = \tau \approx 0.62$. Similarly, $s \to v$ never gets larger than $1 + \tau + \tau^3 + \tau^5 + \cdots = 1 + \tau/(1-\tau^2) = 2$ and $s \to x$ never gets any larger than $\tau + \tau^2 + \tau^3 + \tau^4 + \cdots = \tau/(1-\tau) = \tau^{-1} \approx 1.62$. So the total outflow never gets beyond $\tau + 2 + \tau^{-1} \approx 4.24$, which is much less than 21.

Problem 3 *Let G be a graph whose edges are assigned lengths. Let t be the length of the shortest spanning tree of G. Let s be the length of the shortest path which visits every vertex.*

3.(a) *Show that* $t \leq s$*.*

Let Γ be a union of the edges in the shortest path; it is a connected subgraph of G. Let T be a spanning tree of Γ. Then the length of T is $\leq$ the sum of the lengths of the edges of Γ. This, in turn, is $\leq$ the length of the path. (If the path doubles back on itself, then the sum of the lengths of the edges of Γ may less than the length of the path.) So there is a tree with length $\leq s$.

3.(b) *Show that* $s \leq 2t$*.*

Take a path which explores the tree, starting at some root and transversing every edge twice (one in each direction.) This path has length $2t$, so the optimal path has length $\leq 2t$.

3.(c) *Give an example of a graph* G*, with lengths assigned to edges with the following property: For any spanning tree* T *of* G*, there is some pair of vertices so that the distance from* u *to* v *in* T *is* ≥ 100 *times larger than the distance between* u *and* v *in* G*.*

Take a cycle of 101 vertices.