WORKSHEET 10: COMPOSITION SERIES

Let R be a ring and let M be an R-module.

Definition: A chain of submodules of M is a sequence $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$. We call ℓ the ***length*** of the chain.

Definition: A ***composition series*** is a chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ such that each quotient module M_i/M_{i-1} is simple. We recall that the zero module is **not** considered simple, so $M_i \neq M_{i+1}$ in a composition series.

Problem 10.1. Suppose that there is a positive integer L such that, for any chain $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M$, we have $\ell \leq L$. Show that M has a composition series. (Hint: Consider a chain of maximal length.)

Definition: We say that M has ***finite length*** if M has a composition series.

Problem 10.2. Let M be an R-module which is a finite set. Show that M has finite length.

Problem 10.3. Let k be a field which is contained in R. Suppose that M is finite dimensional as a k-vector space. Show that M has finite length.

The following nonstandard definition will be convenient:

Definition: A ***quasi-composition series*** is a chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$ such that each quotient module M_i/M_{i-1} is either simple or 0.

Problem 10.4. Show that, if M has a quasi-composition series, then M has a composition series.

Problem 10.5. Let $\alpha : A \hookrightarrow B$ be an injective R-module homomorphism, and let $0 = B_0 \subset B_1 \subset \cdots \subset B_b = B$ be a composition series. Show that $\alpha^{-1}(B_0) \subseteq \alpha^{-1}(B_1) \subseteq \cdots \subseteq \alpha^{-1}(B_b)$ is a quasi-composition series.

Problem 10.6. Let $\beta : B \hookrightarrow C$ be an surjective R-module homomorphism, and let $0 = B_0 \subset B_1 \subset \cdots \subset B_b = B$ be a composition series. Show that $\beta(B_0) \subseteq \beta(B_1) \subseteq \cdots \subseteq \beta(B_b)$ is a quasi-composition series.

This, the property of having a composition series passes to submodules and to quotient modules.