WORKSHEET 11: THE JORDAN-HOLDER THEOREM

Definition: A ***short exact sequence of R-modules*** is three R-modules A, B and C, and two R-module homomorphisms $\alpha : A \to B$ and $\beta : B \to C$ such that α is injective, β is surjective and $\operatorname{Im}(\alpha) = \operatorname{Ker}(\beta)$. We write it as $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$.

Throughout the worksheet, let R be a ring and let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules.

Last time, we saw that, if $B_0 \subset B_1 \subset \cdots \subset B_\ell$ is a composition series for B, then $\alpha^{-1}(B_0) \subseteq \alpha^{-1}(B_1) \subseteq \cdots \subseteq \alpha^{-1}(B_\ell)$ is a quasi-composition series for A and $\beta(B_0) \subseteq \beta(B_1) \subseteq \cdots \subseteq \beta(B_\ell)$ is a quasi-composition series for C. The next problem is probably the most technical one:

Problem 11.1. With notation as above, show that **exactly one of the following things** is true:

(1) Either $\alpha^{-1}(B_{i-1}) = \alpha^{-1}(B_i)$ and $\beta(B_i)/\beta(B_i) \cong B_i/B_{i-1}$
(2) or else $\alpha^{-1}(B_i)/\alpha^{-1}(B_i) \cong B_i/B_{i-1}$ and $\beta(B_{i-1}) = \beta(B_i)$.

We are now ready to begin our attack on the Jordan-Holder theorem. We make the following temporary definitions:

Definition: Let M be an R-module of finite length and let $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ be a composition series. Then we define $\ell(M, M_\bullet)$ to be the length m of the composition series $M_\bullet$. For any simple module S, we define $\operatorname{Mult}(S, M, M_\bullet)$ to the number of indices i for which $M_i/M_{i-1} \cong S$.

Theorem (Jordan-Holder): Let M be an R-module of finite length. Suppose that M has two composition series, $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ and $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_n = N$. Then $\ell(M, M_\bullet) = \ell(M, M'_\bullet)$ and, for any simple module S, we have $\operatorname{Mult}(S, M, M_\bullet) = \operatorname{Mult}(S, M, M'_\bullet)$.

In other words, Jordan-Holder shows that $\ell(M)$ and $\operatorname{Mult}(S, M)$ are well-defined quantities.

Problem 11.2. Let $B_\bullet$ be a composition series for B. Define $\tilde{A}_i = \alpha^{-1}(B_i)$ and $\tilde{C}_i = \beta(B_i)$, and let $A_\bullet$ and $C_\bullet$ be the composition series obtained from deleting duplicate elements from $\tilde{A}_\bullet$ and $\tilde{C}_\bullet$. Show that $\ell(B, B_\bullet) = \ell(A, A_\bullet) + \ell(C, C_\bullet)$ and that, for any simple module S, we have $\operatorname{Mult}(S, B, B_\bullet) = \operatorname{Mult}(S, A, A_\bullet) + \operatorname{Mult}(S, C, C_\bullet)$.

Problem 11.3. Show that, if the Jordan-Holder theorem holds for A and C, then it holds for B.

Problem 11.4. Show that the Jordan-Holder theorem holds if the module M is simple.

Problem 11.5. Prove the Jordan-Holder theorem. Hint: Induct on $\min\{\ell : M \text{ has a composition series of length } \ell\}$.