WORKSHEET 12: NOETHERIAN RINGS

Due to the Jordan-Holder thoerem, finite length modules are very well behaved. They make a great subject for study, but unfortunately, many modules we meet naturally are not finite length.

A weaker condition than "finite length" is "finitely generated", which many more modules obey. Over a general ring, finitely generated modules can be very tricky. But, over Noetherian[1] rings, they are not so bad:

Let R be a ring. Consider the following conditions on R.

Condition 1(a): Every left ideal I of the ring R is finitely generated.	Condition 2(a): For any chain of left ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of R, we have $I_r = I_{r+1}$ for all sufficiently large r.	Condition 3(a): Given any nonempty collection $\mathcal{X}$ of left ideals of R, there is some $I \in \mathcal{X}$ which is not properly contained in any other $I' \in \mathcal{X}$.
Condition 1(b): Every left R-submodule M of R^n is finitely generated.	Condition 2(b): For any chain of left R-submodules $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of R^n, we have $M_r = M_{r+1}$ for all sufficiently large r.	Condition 3(b): Given any nonempty collection $\mathcal{X}$ of left R-submodules of R^n, there is some $M \in \mathcal{X}$ which is not properly contained in any other $M' \in \mathcal{X}$.
Condition 1(c): For any finitely generated left R-module S, every left R-submodule M of S is finitely generated.	Condition 2(c): For any finitely generated left R-module S, for any chain of left R-submodules $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of S, we have $M_r = M_{r+1}$ for all sufficiently large r.	Condition 3(c): For any finitely generated left R-module S, given any nonempty collection $\mathcal{X}$ of left R-submodules of S, there is some $M \in \mathcal{X}$ which is not properly contained in any other $M' \in \mathcal{X}$.

Problem 12.1.
Prove all these definitions are equivalent.[2]

Definition: A ring which obeys these conditions is called ***left Noetherian***. A ring which obeys these conditions with "right" in place of "left" is called ***right Noetherian***. A ring which is left and right Noetherian is called ***Noetherian***.

[1]Named for Emmy Noether, German mathematician 1882-1935, who has a decent case for being the greatest algebraist of all time.

[2]If you don't assume the Axiom of Choice, then the conditions in each column are still equivalent to each other, and the implications $3(x) \implies 1(x) \implies 2(x)$ still hold, but I don't know about the reverse implications. However, the use of Choice in showing $2(x) \implies 3(x)$ is very simple.