WORKSHEET 13: UNIQUE FACTORIZATION DOMAINS (UFDS)

Throughout this worksheet, let R be an integral domain.

Definition: Let r be an element of R. We say that r is ***composite*** if r is nonzero and r can be written as a product of two non-units. We say that r is ***irreducible*** if it is neither composite, nor 0, nor a unit.

Thus every element of R is described by precisely one of the adjectives "zero", "unit", "composite", "irreducible".

Definition: Let $p \in R$. We say that p is ***prime*** if pR is a prime ideal and $p \neq 0$.

Problem 13.1. Let p be a non-zero, non-unit. Show that p is prime if and only if, whenever $p|ab$, either $p|a$ or $p|b$.

Problem 13.2. Show that prime elements are irreducible.

Problem 13.3. Let k be a field and let $k[t^2, t^3]$ be the subring of $k[t]$ generated by t^2 and t^3.

(1) Check that t^2 and t^3 are irreducible in $k[t^2, t^3]$.
(2) Show that t^2 and t^3 are not prime in $k[t^2, t^3]$.

Problem 13.4. Consider the subring $\mathbb{Z}[\sqrt{-5}]$ of $\mathbb{C}$.

(1) Show that 2, 3 and $1 \pm \sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$. Hint: Use the complex absolute value.
(2) Show that 2, 3 and $1 \pm \sqrt{-5}$ are not prime in $\mathbb{Z}[\sqrt{-5}]$. Hint: $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.

We want to say that factorizations into prime elements are unique, but factorizations into irreducible elements need not be. In order to do this, we need some vocabulary.

Definition: We define two elements, p and q, of R to be ***associate*** if there is a unit u such that $p = qu$. We define two factorizations $p_1 p_2 \cdots p_m$ and $q_1 q_2 \cdots q_n$ to be ***equivalent*** if $m = n$ and there is a permutation σ in S_n such that p_j is associate to $q_{\sigma(j)}$.

Problem 13.5. Show that any non-zero, non-unit element of R has at most one factorization into **prime** elements, up to equivalence.

Problem 13.6. Give examples, in the rings $k[t^2, t^3]$ and $\mathbb{Z}[\sqrt{-5}]$, of elements with multiple, nonequivalent, factorizations into **irreducible** elements.

Definition: We'll make the following nonstandard definition: We'll say that R ***has factorizations*** if every non-zero, non-unit[1]n R can be written in **at least** one way as a product of irreducibles.

Problem 13.7. Let R have factorizations. Show that the following conditions are equivalent:

(a) All irreducible elements are prime.
(b) Factorizations into irreducibles are unique, up to equivalence.
(c) Every nonzero, nonunit, element has a factorization into prime elements.

Definition: An integral domain which has factorizations and in which the equivalent conditions in Problem 13.7 hold, is called a ***unique factorization domain***, also known as a ***UFD***.

Problem 13.8. Let R be a Noetherian integral domain.

(1) Let r_1, r_2, r_3 ... be a sequence of elements of R such that r_{j+1} divides r_j for all j. Show that, for j sufficiently large, r_j and r_{j+1} are associates.
(2) Show that R has factorizations.

[1]Morally, we should consider the product of the empty set to be 1, so 1 has a factorization into a set of irreducibles, namely the empty set. But trying to get this right would be a notational pain, so we'll just refuse to consider factorizations of units.