WORKSHEET 14: PRINCIPAL IDEAL DOMAINS (PIDS)

Definition: Let R be a commutative ring. An ideal I of R is called ***principal*** if $I = rR$ for some $r \in R$.

Problem 14.1. Show that every ideal in $\mathbb{Z}$ is principal. Do **not** assume unique factorization into primes. (Hint: Take the smallest positive element of the ideal.)

Definition: A ***Principal Ideal Domain*** or ***PID*** is an integral domain in which every ideal is principal.

Problem 14.2. Show that every PID is Noetherian.

Problem 14.3. Let R be a PID. Let u and v be two relatively prime elements of R meaning that, if g divides u and g divides b, then g is a unit. Show that u and v are comaximal, meaning that $uR + vR = R$.

Problem 14.4. Let R be a PID, let p be an irreducible element of R, and let a be any element of R. Show that either p divides a or else p and a are comaximal.

Problem 14.5. Show that, in a PID, irreducible elements are prime.

Problem 14.6. Show that a PID is a UFD.[1]

We note in particular that we have now shown $\mathbb{Z}$ is a UFD.

Problem 14.7. Since PID's are UFD's, we can talk about GCD's in them. Show that, if R is a PID and a and $b \in R$, then $aR + bR = \mathrm{GCD}(a, b)R$.

Problem 14.8. Suppose R is a PID. Show that every nonzero prime ideal in R is a maximal ideal.

We conclude with some fun and useful lemmas about matrices over PID's:

Problem 14.9. Let R be a PID and let x and $y \in R$. Show that there is a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with entries in R and determinant 1 and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{GCD}(x, y) \\ 0 \end{bmatrix}.$$

Problem 14.10. Let R be a PID and let x and $y \in R$. Show that there are 2×2 matrices U and V with entries in R and determinant 1 such that:

$$U \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} V = \begin{bmatrix} \mathrm{GCD}(x, y) & 0 \\ 0 & \mathrm{LCM}(x, y) \end{bmatrix}.$$

Here $\mathrm{LCM}(x, y) := \frac{xy}{\mathrm{GCD}(x,y)}$.

[1] This need not hold without Choice; Hodges, "Lauchli's algebraic closure of Q", *Proceedings of the Cambridge Philosophical Society*, 1976 showed that it is consistent with ZF for there to be a PID in which some elements have no factorization into irreducibles.