WORKSHEET 16: EUCLIDEAN RINGS

Definition: Suppose R is an integral domain. A ***norm*** on R is any function $N\colon R \to \mathbb{Z}_{\geq 0}$. The function N is said to be a ***positive norm*** provided that $N(r) > 0$ for all nonzero r. We call N a ***multiplicative norm*** if $N(ab) = N(a)N(b)$.

Some examples: The normal absolute value on $\mathbb{Z}$ is a positive norm. The norm map $N(a+bi) = a^2+b^2$ on the Gaussian Integers $\mathbb{Z}[i]$ is a positive norm. If k is a field, then we can define a norm on $k[x]$ by $N(p(x)) = \deg p$ for $p \neq 0$ and $N(0) = 0$. [1] We can be a bit more clever and make our norm positive and multiplicative by choosing some positive integer $c \geq 2$ and defining $N(p) = c^{\deg(p)}$ for $p \neq 0$ and $N(0) = 0$.

Definition: An integral domain R is called an ***Euclidean Domain*** provided that there is a positive norm N on R such that for any two elements $a, b \in R$ with $b \neq 0$ there exist q, and $r \in R$ with

$$a = bq + r \text{ and } N(r) < N(b).$$

The element q is called the ***quotient*** and the element r is called the ***remainder*** of the division.

Problem 16.1. Let k be a field. Show that k is Euclidean with respect to the norm that $N(0) = 0$ and $N(x) = 1$ for $x \neq 0$.

Problem 16.2. Let k be a field. Verify that $k[x]$ is Euclidean with respect to the norm $N(p) = c^{\deg(p)}$ discussed at the end of the paragraph above.

Problem 16.3. Let R be an integral domain with positive multiplicative norm N, and let K be its field of fractions. For $\frac{a}{b} \in K$, define $N_K\left(\frac{a}{b}\right) = \frac{N(a)}{N(b)}$.

(1) Show that $N_K(\ )$ is a well defined function $K \to \mathbb{Q}_{\geq 0}$.
(2) Show that R is Euclidean with respect to N if and only if, for each $x \in K$, there is an $q \in R$ with $N_K(x-q) < 1$.

Problem 16.4. Verify that $\mathbb{Z}[i]$ is Euclidean with respect to the norm $N(a+bi) = a^2+b^2$.

Problem 16.5. Show that every Euclidean domain is a PID.

Here are some bonus fun problems about Euclidean domains.

Problem 16.6. Show that $\mathbb{Z}[\sqrt{-2}]$ is Euclidean, with respect to the norm $N(a+b\sqrt{-2}) = a^2+2b^2$.

Problem 16.7. Show that $\mathbb{Z}[\sqrt{-3}]$ is **not** Euclidean, with respect to the norm $N(a+b\sqrt{-3}) = a^2+3b^2$, but that $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is Euclidean with respect to the norm $N\left(\frac{c+d\sqrt{-3}}{2}\right) = \frac{c^2+3d^2}{4}$.

Problem 16.8. Let p be a positive prime integer.

(1) Show that $\mathbb{Z}[i]$ has an ideal π with $\#(\mathbb{Z}[i]/\pi) = p$ if and only if there is a square root of -1 in $\mathbb{Z}/p\mathbb{Z}$.
(2) Show that $\mathbb{Z}[i]$ has a principal ideal $(a+bi)\mathbb{Z}[i]$ with $\mathbb{Z}[i]/(a+bi)\mathbb{Z}[i]$ if and only if p is of the form a^2+b^2.
(3) Conclude the following statement which never mentions the ring $\mathbb{Z}[i]$: A prime p is of the form a^2+b^2 if and only if there is a square root of -1 in $\mathbb{Z}/p\mathbb{Z}$.[2]

Problem 16.9. Let R be a Euclidean domain. Show that there is some nonunit f such that every nonzero residue class in R/fR is represented by a unit of R. Deduce that $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is not Euclidean for any norm function.

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[1] Under various circumstances, it can be reasonable to define the degree of the 0 polynomial to be $-\infty$, 0 or ∞. We do not take a stand on this issue here. Some people define the degree of the 0 polynomial to be -1, but David Speyer sees no justification for this.

[2] The primes p for which this occurs are precisely 2 and the primes which are $1 \bmod 4$. Here is a quick proof: If $p \equiv 1 \bmod 4$, then $-1 \equiv (p-1)! \equiv (-1)^{(p-1)/2}((p-1)/2)!^2 \equiv ((p-1)/2)!^2 \bmod p$. Conversely, if p is odd and $-1 \equiv x^2 \bmod p$ then $(-1)^{(p-1)/2} \equiv x^{p-1} \equiv 1 \bmod p$, so $p \equiv 1 \bmod 4$.