WORKSHEET 19: CLASSIFICATION OF FINITELY GENERATED MODULES OVER A PID

Problem 19.1. Let S be a commutative ring and let M be a finitely generated S-module.

(1) Show that there is a surjection $S^{\oplus m} \twoheadrightarrow M$ for some m.
(2) Suppose that S is Noetherian (for example, every PID is Noetherian). Show that there is a surjection $S^n \twoheadrightarrow \mathrm{Ker}(S^m \to M)$ for some n.
(3) With hypotheses and assumptions as in the previous part, show that there is an $m \times n$ matrix X with $M \cong S^m/XS^n$.

The previous problem shows that every finitely generated S-module is of the form S^m/XS^n for some $m \times n$ matrix X. Now, and **throughout the worksheet, let R be a PID**. We will see how to understand the structure of R^m/XR^n in terms of the Smith normal form of X.

Problem 19.2.
Let $X \in \mathrm{Mat}_{m\times n}(R)$ and let $(d_1, d_2, \ldots, d_{\min(m,n)})$ be the invariant factors of X.

(1) Show that $R^m/XR^n \cong R^{m-\min(m,n)} \oplus \bigoplus_j R/d_jR$.
(2) Show that $\mathrm{Ker}(X) \cong R^{\#\{j : d_j = 0\} + n - \min(m,n)}$.

Problem 19.3. (Classification of modules over a PID: Elementary divisor form) Show that every finitely generated R-module M is of the form $\bigoplus R/d_jR$ for some nonunits $d_1, d_2, \ldots, d_k$ in R with $d_1|d_2|\cdots|d_k$.

Problem 19.4. (Classification of modules over a PID: Prime power form) Show that every finitely generated R-module M is of the form $R^{\oplus r} \oplus \bigoplus R/p_j^{e_j}R$ for some nonnegative integer r, some sequence of prime elements p_j and some sequence of positive integers e_j.

Problems 19.3 and 19.4 each give a list of modules such that every finitely generated R-module M is isomorphic to some module in this list. In for this to be a full classification, we now turn to the problem of checking that these lists do not contain two isomorphic modules, so that we have not listed any isomorphism classes more than once. We'll carry this out for the prime power form.

Problem 19.5. Let q be a prime element of R and let M be an R-module.

(1) Show that R/qR is a field and that, for any $k \geq 0$, that $q^kM/q^{k+1}M$ is an R/qR-vector space.
(2) Let $M = R^{\oplus r} \oplus \bigoplus R/p_j^{e_j}R$ as in Problem 19.4. Give a formula for the dimension of $q^kM/q^{k+1}M$ as an R/qR-vector space in terms of the e_j and r.
(3) Suppose that $R^{\oplus r} \oplus \bigoplus R/p_j^{e_j}R \cong R^{\oplus r'} \oplus \bigoplus R/p_j^{e'_j}R$. Show that $r = r'$ and $e_j = e'_j$.

If you have extra time, do the elementary divisors form as well:

Problem 19.6. Let $d_1, d_2, \ldots, d_k$ and $d'_1, d'_2, \ldots, d'_{k'}$ be nonunits of R with $d_1|d_2|\cdots|d_k$ and $d'_1|d'_2|\cdots|d'_{k'}$, such that $\bigoplus R/d_iR \cong \bigoplus R/d'_iR$. Show that $k = k'$ and d_i is associate to d'_i.