Worksheet 20: Applications of Jordan Normal form and rational canonical form

The point of this section is to give some examples of problems where knowing Jordan Normal form is useful.

Problem 20.1. Let A be a 5×5 complex matrix with minimal polynomial $X^5 - X^3$.

(1) What is the characteristic polynomial of A^2?
(2) What is the minimal polynomial of A^2?

Problem 20.2. In this problem, we investigate square roots of matrices:

(1) Let $g \in \mathrm{GL}_n(\mathbb{C})$. Show that there is an h in $\mathrm{GL}_n(\mathbb{C})$ with $h^2 = g$.
(2) Show that there is no matrix h in $\mathrm{GL}_2(\mathbb{R})$ with $h^2 = \left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right]$.

Problem 20.3. Let k be an algebraically closed field and let A be an $n \times n$ matrix with entries in k. Show that A can be written in the form $D + N$ where D is diagonalizable, N is nilpotent and $DN = ND$. This is called the ***Jordan-Chevalley decomposition*** of A.[1]

Problem 20.4. Let k be an algebraically closed field[2] and let A be an $n \times n$ matrix with entries in k. We define

$$k[A] = \mathrm{Span}_k(1, A, A^2, A^3, A^4, \dots) \subseteq \mathrm{Mat}_{n\times n}(k).$$
$$Z(A) = \{B \in \mathrm{Mat}_{n\times n}(k) : AB = BA\}.$$

(1) Show that $k[A] \subseteq Z(A)$. (I don't recommend Jordan form here.)
(2) Show that the following are equivalent:
   (a) $\dim_k k[A] = n$.
   (b) $\dim_k Z(A) = n$.
   (c) $k[A] = Z(A)$.
   (d) The minimal polynomial of A is the same as the characteristic polynomial of A.
   (e) For each eigenvalue λ of A, there is only one Jordan block of A.

A matrix which obeys the conditions above is called ***regular***.

(3) For any matrix A, show that $\dim k[A] \leq n$.
(4) For any matrix A, show that $\dim Z(A) \geq n$.

Problem 20.5. Let's prove that a real symmetric matrix is diagonalizable!

(1) Let X be an $n \times n$ real matrix and suppose that X is **not** diagonalizable. Prove that there is a two dimensional subspace V of $\mathbb{R}^n$ such that X takes V to itself by a matrix of the form $\left[\begin{smallmatrix} 0 & -c \\ 1 & -b \end{smallmatrix}\right]$ with $b^2 - 4c \leq 0$. (A hint to handle a technical issue: Notice that the matrices $\left[\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & -\lambda^2 \\ 1 & 2\lambda \end{smallmatrix}\right]$ are similar.)
(2) Now suppose that X is symmetric. Let $\cdot$ be the ordinary dot product on $\mathbb{R}^n$. Show that, for any v and $w \in \mathbb{R}^n$, we have $(Xv) \cdot w = v \cdot (Xw)$.
(3) Now suppose that X is symmetric and non-diagonalizable. Let V be the subspace in part (1) and let v, w be a basis for V on which X acts by the matrix $\left[\begin{smallmatrix} 0 & -c \\ 1 & -b \end{smallmatrix}\right]$ with $b^2 - 4c \leq 0$. Show that $w \cdot w + b(v \cdot w) + c(v \cdot v) = 0$.
(4) Deduce a contradiction. Hint: Recall the Cauchy-Schwarz inequality $(v \cdot w)^2 \leq (v \cdot v)(w \cdot w)$.

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[1] The Jordan-Chevalley decomposition is unique, but that is a bit hard for a worksheet; it might occur on a problem set.

[2] In fact, this result is true over any field, except that one needs to refer to generalized Jordan form in (3).(d). I thought that might be a bit too hard for the worksheet though.