WORKSHEET 22: SOME PROBLEMS ABOUT EXTERIOR ALGEBRA

Problem 22.1. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. Expand

$$(e_1 + e_2 + e_3) \wedge (e_1 + 2e_2 + 3e_3)$$

in the basis $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_2 \wedge e_3$ of $\mathbb{R}^3$.

Problem 22.2. Let $L : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. What are the eigenvalues of $\bigwedge^2 L$? Of $\bigwedge^k L$?

Problem 22.3. Let $v_1, v_2, \ldots, v_d$ be vectors in a vector space V. Show that $v_1 \wedge v_2 \wedge \cdots \wedge v_d = 0$ if and only if the v_i are linearly dependent.

Problem 22.4. Let V be a vector space over a field k and let $\eta \in \bigwedge^d V$ for $d > 0$.

(1) Let v be a nonzero vector in V. Show that $v \wedge \eta = 0$ if and only if η can be factored as $v \wedge \theta$ for $\theta \in \bigwedge^{d-1} V$.
(2) More generally, let $U = \{v \in V : v \wedge \eta = 0\}$ and let $u_1, u_2, \ldots, u_k$ be a basis of U. Show that η can be factored as $u_1 \wedge u_2 \wedge \cdots \wedge u_k \wedge \psi$ for some $\psi \in \bigwedge^{d-k} V$.

Problem 22.5. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$.

(1) Show that there is a unique isomorphism $h : \bigwedge^2 \mathbb{R}^3 \to \mathbb{R}^3$ such that, for $v \in \mathbb{R}^3$ and $\eta \in \bigwedge^2 \mathbb{R}^3$, we have $v \wedge \eta = (v \cdot h(\eta)) e_1 \wedge e_2 \wedge e_3$. Here the $\cdot$ is the standard dot product.
(2) The ***cross product*** map $V \times V \to V$ is defined by $v \times w := h(v \wedge w)$. Check that this is the cross product you already know.
(3) Let $g \in \mathrm{SO}(3)$. Show that $gh(\eta) = h(\bigwedge^2(g)\eta)$ and show that $g(u \times v) = g(u) \times g(v)$.

Problem 22.6. Let V be a vector space of dimension n. Let $L : V \to V$ be a linear map; we will also write L for the matrix of L. Recall that the adjugate matrix, $\mathrm{Adj}(L)$, is the matrix whose (i, j) entry is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ minor of L where we delete row j and column i. For example,

$$\mathrm{Adj} \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} vz - wy & -(sz - ty) & sw - tv \\ -(uz - wx) & rz - tx & -(rw - tu) \\ uy - vx & -(ry - sx) & rv - su \end{bmatrix}.$$

(1) What is the relation between $\mathrm{Adj}(L)$ and $\bigwedge^{n-1}(L)$?
(2) For any $v \in V$ and $\eta \in \bigwedge^{n-1}(V)$, show that $L(v) \wedge \bigwedge^{n-1}(L)(\eta) = (\det L)(v \wedge \eta)$.
(3) Show that $L\ \mathrm{Adj}(L) = (\det L)\mathrm{Id}_n$.