WORKSHEET 2: MODULES

Groups are meant to act on sets. Similarly, rings are meant to act on abelian groups.

Definition: Suppose R is a ring. A ***left R-module*** is a set M with two operations:

- $+\colon M \times M \to M$ (called ***addition***) and
- $*\colon R \times M \to M$ (called ***scalar multiplication***)

and an element 0_M satisfying the following axioms:

M1: $(M, +, 0_M)$ is an abelian group,
M2: $(r+s)*m = r*m + s*m$ for all $r, s \in R$ and $m \in M$
M3: $(rs)*m = r*(s*m)$ for all $r, s \in R$ and $m \in M$
M4: $r*(m+n) = r*m + r*n$ for all $r \in R$ and $m, n \in M$
M5: $1_R * m = m$ for all $m \in M$.[1]

"***M is an R-module***" will mean "M is a left R-module".

The map $*\colon R \times M \to M$ is called an ***action*** of R on M and the elements of R are often called ***scalars***.

Problem 2.1. Show that $\mathbb{Z}^n$ is a left-$\mathrm{Mat}_{n\times n}(\mathbb{Z})$-module by having $X \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ act on $\mathbb{Z}^n$ by taking $v \in \mathbb{Z}^n$ to Xv.

Definition. Suppose R is a ring and M and N are R-modules. A function $g\colon M \to N$ is called an ***R-module homomorphism*** provided that

- g is a group homomorphism and
- $g(rm) = rg(m)$ for all $r \in R$ and $m \in M$.

The set of R-module homomorphisms from M to N is denoted $\mathrm{Hom}_R(M, N)$. We set $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ and call $\mathrm{End}_R(M)$ the ***endomorphism ring of*** M.

Problem 2.2. Suppose R is a commutative ring and M is an R-module. Show that there is a "natural" map of rings $R \to \mathrm{End}_R(M)$. What if R is not commutative?

Definition. Suppose R is a ring and M and N are R-modules. The ***direct sum*** of M and N, written $M \oplus N$, is the R-module defined as follows: An element of $M \oplus N$ is an ordered pair (m, n) with $m \in M$ and $n \in N$. We have $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$ and $r(m, n) = (rm, rn)$.

Problem 2.3. Check that $M \oplus N$ is an R-module.

Problem 2.4. Let M_1, M_2, M, N_1, N_2 and N be R-modules. Show that $\mathrm{Hom}_R(M_1 \oplus M_2, N) \cong \mathrm{Hom}_R(M_1, N) \times \mathrm{Hom}_R(M_2, N)$ and $\mathrm{Hom}_R(M, N_1 \oplus N_2) \cong \mathrm{Hom}_R(M, N_1) \times \mathrm{Hom}_R(M, N_2)$ as abelian groups.

Problem 2.5. Let $L_1, L_2, \ldots, L_p$, $M_1, M_2, \ldots, M_q$ and $N_1, N_2, \ldots, N_r$ be R-modules, and let $L = \bigoplus L_i$, $M = \bigoplus M_j$ and $N = \bigoplus N_k$. Describe a way to write elements of $\mathrm{Hom}_R(L, M)$, $\mathrm{Hom}_R(M, N)$ and $\mathrm{Hom}_R(L, N)$ as matrices, so that the composition map $\mathrm{Hom}_R(L, M) \times \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(L, N)$ corresponds to matrix multiplication.

[1] As you might guess, some people do not impose this last condition.