WORKSHEET 3: IDEALS

Definition: Suppose R is a ring. A subset $I \subset R$ is called a ***left ideal*** provided that

I1: $(I, +)$ is a subgroup of $(R, +)$; and

I2: for all $r \in R$ we have $rI \subset I$, that is $rx \in I$ for all $x \in I$.

It is called a ***right ideal*** provided that

I1: $(I, +)$ is a subgroup of $(R, +)$; and

I2: for all $r \in R$ we have $Ir \subset I$, that is $yr \in I$ for all $y \in I$.

A subset of R that is both a left and right ideal is called a ***two-sided ideal***.

If R is commutative, then "left ideal", "right ideal" and "two-sided ideal" are the same, and we will simply write ***ideal***. [1]

Problem 3.1. Show that if A and B are ideals, then $A + B := \{a + b : \ a \in A, \ b \in B\}$ is also an ideal.

Problem 3.2. Fix $n \geq 2$. Let I be the subset of $R = \mathrm{Mat}_{n\times n}(\mathbb{Q})$ consisting of matrices with nonzero entries only in the first row. Is I a left ideal? Is it a right ideal?

Problem 3.3. Suppose R and S are rings and $\varphi \in \mathrm{Hom}(R, S)$. Show that $\ker(\varphi)$ is a two-sided ideal of R.

Problem 3.4. Let R be a ring and let I be a left ideal. Since I and R are abelian groups with respect to $+_R$, we can form the quotient group R/I. Show that R/I has a natural structure as a left R-module.

Problem 3.5. Let R be a ring and let I be a two sided ideal. Show that R/I has a natural ring structure.

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[1] In this course, we will not use the word "ideal" in a non-commutative ring without saying whether it is a left ideal, right ideal or two-sided ideal. If you see a source using "ideal" by itself in a non-commutative setting, it probably means "two-sided ideal", but Prof. Speyer recommends being clearer and not using the word "ideal" by itself in this context.