Worksheet 4: Integral domains

Definition: A commutative ring R is called an ***integral domain*** if:

ID1: Whenever $xy = 0$ in R, we have either $x = 0$ or $y = 0$ and
ID2: The ring R is not the zero ring.

Integral domains are similar to fields, but not as nice. The next problems explore the relationship.

Problem 4.1. Show that a field is an integral domain.

Problem 4.2. Show that $\mathbb{Z}$ is an integral domain but not a field.

Problem 4.3. Show that $k[x]$ is an integral domain but not a field, where k is a field.

Problem 4.4. Let R be a nonzero commutative ring.

(1) Show that R is an integral domain if and only if, for all $x \neq 0$ in R, the map $y \mapsto xy$ is injective.
(2) Show that R is a field if and only if, for all $x \neq 0$ in R, the map $y \mapsto xy$ is bijective.

Problem 4.5. Let R be an integral domain and suppose that $\#(R)$ is finite. Show that R is a field.

Problem 4.6. Let R be an integral domain and let k be a subring of R which is a field, such that R is finite dimensional as a k-vector space. Show that R is a field.

Every integral domain R embeds in a natural field, known as the ***field of fractions of*** R and denoted $\mathrm{Frac}(R)$.

Definition: Let R be an integral domain. Define X to be the set of pairs (p, q) in R^2 with $q \neq 0$. Define an equivalence relation $\sim$ on X by

$$(p_1, q_1) \sim (p_2, q_2) \text{ if and only if } p_1 q_2 = p_2 q_1.$$

We will denote an element of $X/\sim$ as p/q or $\frac{p}{q}$. We define addition and multiplication on $X/\sim$ by:

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \qquad \frac{p_1}{q_1} * \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2}.$$

We denote this field $\mathrm{Frac}(R)$.

Problem 4.7. Verify that $\sim$ is an equivalence relation on X.

Problem 4.8. Verify that $X/\sim$ is a field under the operations $+$ and $*$ on $X/\sim$.

At this point, we can see why it is a good idea to define $\{0\}$ **not** to be an integral domain: If we try these definitions with $R = \{0\}$, then $X = \emptyset$, so $\mathrm{Frac}(R)$ would be $\emptyset$ and, in particular, would not have additive or multiplicative identities.