WORKSHEET 5: PRIME AND MAXIMAL IDEALS

Definition: Suppose R is a commutative ring. An ideal P of R is called ***prime*** if,

P1: for all a and $b \in R$, if $ab \in P$ then $a \in P$ or $b \in P$.
P2: The ideal P is not all of R.

Problem 5.1. Let R be a commutative ring; let I be an ideal of R. Show that I is prime iff R/I is an integral domain.

Problem 5.2. For which integers n is $n\mathbb{Z}$ prime? You may assume uniqueness of prime factorization for this question. [1]

Definition: Suppose R is a commutative ring. An ideal $\mathfrak{m}$ of R is called ***maximal*** if:

M1: For all a in R, if $a \notin \mathfrak{m}$ then there is some $b \in R$ such that $ab \equiv 1 \bmod \mathfrak{m}$.
M2: The ideal $\mathfrak{m}$ is not all of R.

Problem 5.3. Let R be a commutative ring and let I be an ideal of R. Show that I is maximal and only if R/I is a field.

Problem 5.4. Show that a maximal ideal is prime.

Problem 5.5. Show that an ideal $I \subsetneq R$ is maximal if and only there does not exist an ideal J with $I \subsetneq J \subsetneq R$.

Problem (5.5) is the motivation for the word "maximal". Using Zorn's lemma, and Problem (5.5), it is easy to show that every ideal in a nonzero commutative ring is contained in a maximal ideal.

Problem 5.6. Let $R = \mathbb{R}[x, y]$. Show that yR is prime but not maximal.

Problem 5.7. Let R be a commutative ring and let P be a prime ideal. Suppose that R/P is finite. Show that P is maximal.

Problem 5.8. What are the maximal ideals of $\mathbb{Z}$?

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[1] Pretty soon, we will discuss unique factorization in commutative rings in general. At that point, we will prove it for $\mathbb{Z}$ (and many other rings). The careful student can check that there is no circularity; the problems where I permit you to use it now will not feed into our proof then.