WORKSHEET 6: PRODUCTS OF RINGS AND MODULES

Recall that if A and B are sets, then the product of A and B is the set $A \times B = \{(a, b) \mid a \in A, b \in B\}$. This can be extended to a product of any number of sets. If R and S are rings, then we want the product $R \times S$ to be more than just a set – we want it to be a ring. To make this happen we define addition and multiplication as follows

- $(r, s) + (r', s') = (r + r', s + s')$ for all $(r, s), (r', s') \in R \times S$ and
- $(r, s) * (r', s') = (r * r', s * s')$ for all $(r, s), (r', s') \in R \times S$.

Problem 6.1. Show that $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/15\mathbb{Z}$ are isomorphic as rings.

Problem 6.2. Are there natural ring homomorphisms $R \to R \times S$ and $S \to R \times S$? Are there natural ring homomorphisms $R \times S \to R$ and $R \times S \to S$?

Problem 6.3. Let R and S be rings and let M and N be an R-module and an S-module respectively. Explain how to put an $(R \times S)$-module structure on the abelian group $M \times N$.

Every $(R \times S)$-module breaks up as in Problem 6.3, as the next problem explains.

Problem 6.4. Let R and S be rings. Write e for the element $(1, 0) \in R \times S$. Let M be an $R \times S$ module.

(1) Show that $M = eM \oplus (1 - e)M$.
(2) Show how to equip eM with the structure of an R-module, and $(1 - e)M$ with the structure of an S-module, so that $M \cong eM \times (1 - e)M$ (in the sense of Problem 6.3 .)