WORKSHEET 9: SIMPLE MODULES

Definition: Let R be a ring and let S be a (left) R-module. The module S is called ***simple*** if $S \neq 0$ and the only R-submodules of S are (0) and S.

Problem 9.1. Let R be a ring, let S be a simple R-module, and let M be any R-module.

(1) Let $\alpha : S \to M$ be an R-module homomorphism. Show that α is either injective or 0.
(2) Let $\beta : M \to S$ be an R-module homomorphism. Show that β is either surjective or 0.

Problem 9.2. Let R be a ring and let I be a left ideal. Show that R/I is simple and if and only if there are no left ideals J with $I \subsetneq J \subsetneq R$. Such a left ideal is called a ***maximal left ideal***.

Problem 9.3. If R is a commutative ring, show that this notion of "maximal left ideal" coincides with the notion of "maximal ideal" we have defined before.

Problem 9.4. If the module S is simple, and x is any nonzero element of S, show that $S = Rx$.

Problem 9.5. In any module M, if there is an element x such that $M = Rx$, show that there is a left ideal I of R such that $M \cong R/I$.

Thus, we have shown that the simple R modules are precisely the R-modules of the form R/I for I a maximal left ideal.

Problem 9.6. (**Schur's Lemma**) Let M be a simple R-module. Let $\phi : M \to M$ be an R-module homomorphism. Show that either $\phi = 0$ or else ϕ is invertible.

Schur's Lemma is the first of many results which will relate a property of a module to a property of its endomorphism ring.