WORKSHEET B: RATIONAL CANONICAL FORM OF A MATRIX

Problem B.1. Let k be a field. Make sure everyone in your group remembers how to do the old homework problem: Give an equivalence between (1) $k[t]$-modules which are finite dimensional as k-vector spaces and (2) pairs (V, T) where V is a finite dimensional k-vector space and $T : V \to V$ is a k-linear map.

Problem B.2. Let k be a field and let M_1 and M_2 be $k[t]$-modules which are finite dimensional as k-vector spaces, corresponding to (V_1, T_1) and (V_2, T_2). What is the pair corresponding to $M_1 \oplus M_2$?

Let k be a field and let $f = x^d + f_{d-1}x^{d-1} + \cdots + f_0$ be a monic polynomial with coefficients in k. We define the ***companion matrix*** of f by

$$\mathcal{C}(f) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -f_0 \\ 1 & 0 & 0 & \cdots & 0 & -f_1 \\ 0 & 1 & 0 & \cdots & 0 & -f_2 \\ 0 & 0 & 1 & \cdots & 0 & -f_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -f_{d-1} \end{bmatrix}$$

Problem B.3. Show that $k[x]/f(x)k[x]$ corresponds to the pair $(k^d, \mathcal{C}(f))$.

An $n \times n$ matrix with entries in k is said to be in ***rational***[1] ***canonical form*** if it is a block matrix of the form

$$\begin{bmatrix} \boxed{\mathcal{C}(f_1)} & & & \\ & \boxed{\mathcal{C}(f_2)} & & \\ & & \ddots & \\ & & & \boxed{\mathcal{C}(f_k)} \end{bmatrix}$$

for some monic polynomials $f_1(x)$, $f_2(x)$, …, $f_k(x)$ with $f_1 | f_2 | \cdots | f_k$.

Problem B.4. (**The rational canonical form theorem**) Let V be a finite dimensional k-vector space and let $T : V \to V$ be a k-linear map. Show that there is a basis of V in which T is given by a matrix in rational canonical form, and that the polynomials f_1, f_2, …, f_k are uniquely determined by (V, T).

Problem B.5. Describe the characteristic polynomial of T in terms of f_1, f_2, …, f_k.

Problem B.6. The ***minimal polynomial*** of T is the monic polynomial $g(t) \in k[t]$ of lowest degree such that $g(T) = 0$. Describe the minimal polynomial of T in terms of f_1, f_2, …, f_k.

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[1] The word "rational" is because we can put matrices into rational canonical form while staying in the same ground field, unlike Jordan-canonical form where need to pass to a larger field. It does not indicate that the notion is special to the field $\mathbb{Q}$.