WORKSHEET C: JORDAN AND GENERALIZED JORDAN FORM OF A MATRIX

Let λ be an element of k. We[1] define the ***Jordan block*** by

$$J_n(\lambda) = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda \end{bmatrix}$$

Problem C.1. Show that $(x-\lambda)^{n-1}, (x-\lambda)^{n-2}, \ldots, (x-\lambda), 1$ is a basis for $k[x]/(x-\lambda)^n k[x]$ and show that multiplication by x, in this basis, is given by the matrix $J_n(\lambda)$.

A matrix is said to be in ***Jordan normal form*** if it is a block matrix whose blocks are Jordan blocks.

Problem C.2. (**The Jordan normal form theorem**) Suppose that the field k is algebraically closed. Show that each $n \times n$ matrix with entries in k is similar to a matrix in Jordan normal form, and that the Jordan normal form is unique up to reordering blocks.

Let $f = x^d + f_{d-1}x^{d-1} + \cdots + f_1 + 0$ be a monic polynomial with coefficients in k. Let U_d be the $d \times d$ matrix with a 1 in the upper-right corner and all other entries 0. Define the ***generalized Jordan block*** $J_n(f(x))$ to be the $(dn) \times (dn)$ block matrix

$$J_n(f) = \begin{bmatrix} \mathcal{C}(f) & 0 & 0 & \cdots & 0 \\ U_d & \mathcal{C}(f) & 0 & \cdots & 0 \\ 0 & U_d & \mathcal{C}(f) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & U_d & \mathcal{C}(f) \end{bmatrix}$$

Problem C.3. Show that $\{x^i f(x)^j : 0 \leq i < d,\ 0 \leq j < n\}$ is a basis for $k[x]/f(x)^n k[x]$.

Problem C.4. Show that multiplication by x in the above basis is given by the matrix $J_n(f(x))$.

Define a matrix to be in ***generalized Jordan normal form*** if it is a block diagonal matrix where each block is of the form $J_{n_i}(p_i(x))$ and the polynomials $p_i(x)$ are irreducible.

Problem C.5. Show that each $n \times n$ matrix with entries in k is similar to a matrix in generalized Jordan normal form, and that the generalized Jordan normal form is unique up to reordering blocks.

[1]The more standard choice is to take $J_n(\lambda)$ to be the transpose of this. The choice given here is more compatible with the standard choices used to define rational canonical form, so we will adopt it. There is no important difference between these conventions.