WORKSHEET E: TENSOR ALGEBRAS, SYMMETRIC AND EXTERIOR ALGEBRAS

Let k be a field and let V be a vector space over k. There is a natural isomorphism $(V \otimes V) \otimes V \cong V \otimes (V \otimes V)$ and similarly for higher tensor powers. We therefore write $V^{\otimes n}$ for the n-fold tensor product of V with itself and write elements of $V^{\otimes n}$ as $\sum c_{j_1 j_2 \cdots j_n} v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_n}$ without parentheses. We define $V^{\otimes 0}$ to be k.

We define the ***tensor algebra*** $T(V)$ to be $\bigoplus_d V^{\otimes d}$.

Problem E.1. Show that $T(V)$ has a unique ring structure where the product of $\sigma \in V^{\otimes s}$ and $\tau \in V^{\otimes t}$ is $\sigma \otimes \tau \in V^{\otimes (s+t)}$.

Problem E.2.
Let $L : V \to W$ be a linear map. Show that there is a unique map of rings $T(L) : T(V) \to T(W)$ with $T(L)(v) = L(v)$ for $v \in V$.

We define the symmetric algebra $\mathrm{Sym}^\bullet(V)$ to be the quotient of $T(V)$ by the 2-sided ideal generated by all tensors of the form $v \otimes w - w \otimes v$.

Problem E.3. Show that $\mathrm{Sym}^\bullet(V)$ is a commutative ring.

Problem E.4. Show that $\mathrm{Sym}^\bullet(V)$ breaks up as a direct sum $\bigoplus_{d=0}^{\infty} \mathrm{Sym}^d(V)$ where $\mathrm{Sym}^d(V)$ is a quotient of $V^{\otimes d}$.

Problem E.5. Let $x_1, x_2, \ldots, x_n$ be a basis of V. Show that $\{x_{i_1} x_{i_2} \cdots x_{i_d} \ : \ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n\}$ is a basis of $\mathrm{Sym}^d(V)$. Show that $\mathrm{Sym}^\bullet(V) \cong k[x_1, \ldots, x_n]$.

We define the exterior algebra, $\bigwedge^\bullet(V)$ to be the quotient of $T(V)$ by the two sided ideal generated by $v \otimes v$ for all $v \in V$. The multiplication in $\bigwedge^\bullet(V)$ is generally denoted $\wedge$.

Problem E.6. Show that, for v and $w \in V$, we have $v \wedge w = -w \wedge v$.

Problem E.7. Show that $\bigwedge^\bullet(V)$ breaks up as a direct sum $\bigoplus_{d=0}^{\infty} \bigwedge^d(V)$ where $\bigwedge^d(V)$ is a quotient of $V^{\otimes d}$.

Problem E.8. Let $e_1, e_2, \ldots, e_n$ be a basis of V. Show that $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} \ : \ 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$ is a basis of $\bigwedge^d(V)$.

Problem E.9. Let $v_1, v_2, \ldots, v_d \in V$. Show that $v_1 \wedge v_2 \wedge \cdots \wedge v_d = 0$ if and only if $v_1, v_2, \ldots, v_d$ are linearly dependent.

We now consider the effect of these constructions on linear maps. Let V and W be k-vector spaces and $L : V \to W$ a linear map.

Problem E.10. Show that there are unique ring maps $\mathrm{Sym}^\bullet(L) : \mathrm{Sym}^\bullet(V) \to \mathrm{Sym}^\bullet(W)$ and $\bigwedge^\bullet(L) : \bigwedge^\bullet(V) \to \bigwedge^\bullet(W)$ with $\mathrm{Sym}^\bullet(L)(v) = L(v)$ and $\bigwedge^\bullet(L)(v) = L(v)$ for $v \in V$.

Problem E.11. Let $L : k^3 \to k^3$ be given by the matrix $\left[\begin{smallmatrix} r & s & t \\ u & v & w \\ x & y & z \end{smallmatrix}\right]$. Compute the matrix of $\bigwedge^2(L) : \bigwedge^2(k^3) \to \bigwedge^2(k^3)$.

Problem E.12. Let $L : k^2 \to k^2$ be given by the matrix $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$. Compute the matrix of $\mathrm{Sym}^2(L) : \mathrm{Sym}^2(k^2) \to \mathrm{Sym}^2(k^2)$.

Problem E.13. Show that $\bigwedge^d(L \circ M) = \bigwedge^d(L) \circ \bigwedge^d(M)$ and $\mathrm{Sym}^d(L \circ M) = \mathrm{Sym}^d(L) \circ \mathrm{Sym}^d(M)$.

Given an $m \times n$ matrix X with entries in k, and subsets $I \subseteq \{1, 2, \ldots, m\}$ and $J \subseteq \{1, 2, \ldots, n\}$ of the same size, define $\Delta_{IJ}(X)$ to be the determinant of the square submatrix of X using rows I and columns J.

Problem E.14. Prove the Cauchy-Binet formula: Let X and Y be $a \times b$ and $b \times c$ matrices with entries in k and let I and K be subsets of $\{1, 2, \ldots, a\}$ and $\{1, 2, \ldots, c\}$ with $|I| = |J| = q$. Then

$$\Delta_{IK}(XY) = \sum_{\substack{J \subseteq \{1, 2, \ldots, b\} \\ |J| = q}} \Delta_{IJ}(X) \Delta_{JK}(Y).$$