WORKSHEET G: SYMMETRIC BILINEAR FORMS

Let k be a field, let V be a finite dimensional vector space over k and let $B : V \times V \to k$ be a k-bilinear pairing. Recall that, given a basis $v_1, v_2, \ldots, v_n$ of V, we encode B in a Gram matrix G with $G_{ij} = B(v_i, v_j)$, and that B is symmetric if and only if G is. Changing bases of V modifies the Gram matrix by $G \mapsto SGS^T$ for invertible S. It is natural to ask how nice we can make the matrix G by action of this kind.

To simplify our results, assume that k does not have characteristic 2.

Problem G.1. Suppose that B is a symmetric bilinear form. Show that there is a basis of V for which the Gram matrix of B is diagonal. (Hint: If $B \neq 0$, use Problem F.3 to find a vector v with $B(v, v) \neq 0$, then consider the decomposition $V = kv \oplus (kv)^\perp$.)

Problem G.2. Let G be a symmetric matrix with entries in k. Show that there is an invertible matrix S and a diagonal matrix D such that $G = SDS^T$.

Problem G.3. Let $k = \mathbb{Q}$. Carry out the procedure in the previous problems for

(1) $G = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

(2) $G = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

This immediately raises the question, given two diagonal matrices $\mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)$, when are the bilinear forms $\vec{x}^T \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)\vec{y}$ and $\vec{x}^T \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)\vec{y}$ equivalent up to a change of basis? For a general field, this is a very hard question. However, we can say some things.

Problem G.4. Suppose that there are nonzero scalars γ_i in k with $\alpha_i = \gamma_i^2 \beta_i$. Show that the $\vec{x}^T \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)\vec{y}$ and $\vec{x}^T \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)\vec{y}$ are equivalent.

Problem G.5. Show that the bilinear forms $B(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]) = x_1 x_2 + y_1 y_2$ and $C(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]) = 5x_1 x_2 + 5y_1 y_2$ are related by a change of basis in $\mathbb{Q}^2$, even though 5 is not square in $\mathbb{Q}$.

Problem G.6. Let $k = \mathbb{R}$. Show that any bilinear form over $\mathbb{R}$ can be represented by a diagonal matrix whose entries lie in $\{-1, 0, 1\}$.