WORKSHEET H: SYMMETRIC BILINEAR FORMS OVER $\mathbb{R}$

Let B be a symmetric bilinear form on a vector space W over $\mathbb{R}$. We say that B is

- ***Positive definite*** if $B(w,w) > 0$ for all nonzero $w \in W$.
- ***Positive semidefinite*** if $B(w,w) \geq 0$ for all $w \in W$.
- ***Negative definite*** if $B(w,w) < 0$ for all nonzero $w \in W$.
- ***Negative semidefinite*** if $B(w,w) \leq 0$ for all $w \in W$.

Recall that we showed in Problem G.6 that a symmetric bilinear form over $\mathbb{R}$ can always be represented by a diagonal matrix whose entries lie in $\{-1, 0, 1\}$.

Problem H.1. Let B be a symmetric bilinear form which can be represented by the diagonal matrix

$$\mathrm{diag}(\overbrace{1,1,\ldots,1}^{n_+},\overbrace{0,0,\ldots,0}^{n_0},\overbrace{-1,-1,\ldots,-1}^{n_-}).$$

(1) Show that n_+ is the dimension of the largest subspace L of V such that B restricted to L is positive definite.
(2) Show that $n_+ + n_0$ is the dimension of the largest subspace L of V such that B restricted to L is positive semidefinite.
(3) Show that n_- is the dimension of the largest subspace L of V such that B restricted to L is negative definite.
(4) Show that $n_- + n_0$ is the dimension of the largest subspace L of V such that B restricted to L is negative semidefinite.

Problem H.2. Let B be a symmetric bilinear form. Suppose that B can be represented (in two different bases) by the diagonal matrices

$$\mathrm{diag}(\overbrace{1,1,\ldots,1}^{m_+},\overbrace{0,0,\ldots,0}^{m_0},\overbrace{-1,-1,\ldots,-1}^{m_-}) \text{ and } \mathrm{diag}(\overbrace{1,1,\ldots,1}^{n_+},\overbrace{0,0,\ldots,0}^{n_0},\overbrace{-1,-1,\ldots,-1}^{n_-}).$$

Show that $(m_+, m_0, m_-) = (n_+, n_0, n_-)$.

The word ***signature*** is used to refer to something like the triple (n_+, n_0, n_-). Unfortunately, sources disagree as to exactly what the signature is. Various sources will say that the signature is (n_+, n_0, n_-), (n_+, n_-, n_0), (n_+, n_-) or $n_+ - n_-$. In this course, we'll adopt the convention that the signature is (n_+, n_0, n_-). If G is a symmetric real matrix, we will use the term ***signature of*** G to refer to the signature of the bilinear form $B(x,y) = x^T G y$.

Problem H.3.
Let G be a real symmetric $n \times n$ matrix with signature (n_+, n_0, n_-). If $n_0 > 0$, show that $\det G = 0$. If $n_0 = 0$, show that $\det G$ is nonzero with sign $(-1)^{n_-}$.

Problem H.4. Let G be a real symmetric $n \times n$ matrix with signature (n_+, n_0, n_-). Let G' be the upper left symmetric $(n-1) \times (n-1)$ submatrix of G. Show that the signature of G' is one of $(n_+ - 1, n_0 + 1, n_- - 1)$, $(n_+ - 1, n_0, n_-)$, $(n_+, n_0, n_- - 1)$, $(n_+, n_0 - 1, n_-)$. Hint: Use Problem H.1.

Problem H.5. Let G be a real symmetric matrix and let G_k be the $k \times k$ upper left submatrix of G. Assume that $\det G_k \neq 0$ for $1 \leq k \neq n$. Show that the signature of G is $(n-q, 0, q)$ where q is the number of k for which $\det G_{k-1}$ and $\det G_k$ have opposite signs. Here we formally define $\det G_0 = 1$.

Problem H.6. (**Sylvester's criterion**) Let G be a real symmetric matrix and define G_k as above. Show that G is positive definite if and only if all the $\det G_k$ are > 0. (In other words, we no longer have to take $\det G_k \neq 0$ as an assumption.)