PROBLEM SET ONE: DUE SEPTEMBER 15

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 1.

Problem 2. Write up complete solutions to **three** of the following problems from class:
1.5, 2.5, 3.5, 3.6, 4.6, 4.7, 5.1, 5.3

Problem 3. Let K be a field and let R be a subring of K. Let S be a nonempty subset of R, closed under multiplication and not containing 0. Let $S^{-1}R$ be the set of elements in K which can be written as $\frac{a}{b}$ with $a \in R$ and $b \in S$. Show that $S^{-1}R$ is a subring of K.

Problem 4. Let k be a field. Show that the only ideals of k are (0) and k.

Problem 5. Let k be a field.

(1) Describe all left ideals in the ring $\mathrm{Mat}_{n\times n}(k)$ of $n \times n$ matrices with entries in k. Hint: Row reduction.
(2) Show that the only two-sided ideals of $\mathrm{Mat}_{n\times n}(k)$ are (0) and $\mathrm{Mat}_{n\times n}(k)$.

Problem 6. Suppose k is a field and let $R = k[t]$. Show that R-modules are "the same" as k-vector spaces V with a k-linear endomorphism $T : V \to V$. This question can be interpreted in two ways:

- (for those who don't know what categories are) Give a bijection between isomorphism classes of R-modules and isomorphism classes of pairs (V, T); this includes defining when (V_1, T_1) and (V_2, T_2) are isomorphic.
- (for those who know what categories are) Define the category of R-modules and the category of pairs (V, T), and give an equivalence between them.

Problem 7. A follow up to Problem 6:

(1) Let M be the $\mathbb{R}[t]$-module $\mathbb{R}[t]/(t^3 - 2)\mathbb{R}[t]$. Give an explicit 3×3 matrix for the corresponding T.
(2) Do $(\mathbb{R}^2, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$ and $(\mathbb{R}^2, \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right])$ correspond to isomorphic $\mathbb{R}[t]$ modules or not?

Problem 8. Suppose R is a ring. An element $e \in R$ is called ***idempotent*** if $e^2 = e$.

(1) Give an example of an idempotent, other than 0 and 1, in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.
(2) Give an example of an idempotent, other than 0 and 1, in $\mathbb{Z}/15\mathbb{Z}$.
(3) Let e be an idempotent of R and let $eRe = \{ere : r \in R\}$. Show that eRe is a ring, with respect to the addition and multiplication operations of R, where $0_{eRe} = 0_R$ and $1_{ere} = e$.

Problem 9. Let R be a ring. For $r \in R$ and $1 \le i \ne j \le n$ define the $n \times n$ matrix $E(i, j, r)$ by

$$E(i,j,r)_{k\ell} = \begin{cases} 1 & \text{if } k = \ell, \\ r & \text{if } k = i \text{ and } j = \ell, \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $E(i, j, r)$ is known as an ***elementary matrix***.

(1) Suppose X is an $m \times n$ matrix. What is the effect of right multiplication by $E(i, j, r)$ on X? Suppose Y is an $n \times m$ matrix. What is the effect of left multiplication by $E(i, j, r)$ on Y? What is the inverse of $E(i, j, r)$?
(2) Show that the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is a product of 2×2 elementary matrices.
(3) Let u be a unit of R. Show that the matrix $\left[\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right]$ is a product of 2×2 elementary matrices.