PROBLEM SET 10: DUE DECEMBER 3

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 10. **Both questions will be about tensor products.**

Problem 2. Let V be a vector space over a field k and let r be a positive integer. Let S_n be the group of permutations of $\{1, 2, \dots, r\}$ and let $\epsilon : S_r \to \{\pm 1\}$ be the sign map.

(1) Show that there is a well defined map $\alpha : V^{\otimes r} \to V^{\otimes r}$ such that

$$\alpha(v_1 \otimes v_2 \otimes \cdots \otimes v_r) = \sum_{\sigma \in S_r} \epsilon(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(r)}.$$

(2) Let $\dim V = n$. Compute the rank of the linear map α.

Problem 3. Let k be a field and let V and W be k-vector spaces. Let $V^\vee$ be the dual vector space to V. This problem discusses the relation between $V^\vee \otimes_k W$ and $\mathrm{Hom}(V, W)$.

(1) Show that there is a map $\phi : V^\vee \otimes_k W \to \mathrm{Hom}_k(V, W)$ such that $\phi(\lambda \otimes w)(v) = \lambda(v)w$.
(2) Show that the image of ϕ is precisely the linear maps $V \to W$ of finite rank. In particular, if $\dim V$ or $\dim W < \infty$, show that ϕ is surjective.
(3) Show that every element of $V^\vee \otimes_k W$ can be represented in the form $\sum_{j=1}^n \lambda_j \otimes w_j$ where w_1, w_2, $\dots$, w_n are linearly independent.
(4) Show that ϕ is always injective. Hint: Write an element of the kernel as in Part (3).

Problem 4. Let R be a PID which is not a field and let M be a finite length R-module (so M has no free summand). Let $\mathrm{End}(M)$ be $\mathrm{Hom}_R(M, M)$, considered as an R-module in the obvious way. Let $\mathrm{Ann}(M) = \{r \in R : rm = 0 \text{ for all } m \in M\}$. Recall that we write $\ell(M)$ for the length of a finite length R-module M, and that you computed on Problem Set 8 that $\ell\left(\bigoplus R/p_i^{f_i}R\right) = \sum f_i$.

(1) Show that $\ell(R/\mathrm{Ann}(M)R) \leq \ell(M)$.
(2) Show that $\ell(M) \leq \ell(\mathrm{End}(M))$.
(3) Show that the following are equivalent:
   (a) We have equality in part (1).
   (b) We have equality in part (2).
   (c) When we write M in prime power form as $\bigoplus R/p_i^{e_i}R$, all the p_i are non-associate.
   (d) $M \cong R/dR$ for some nonzero $d \in R$.

Now, let $R = k[T]$, so M is a k-vector space. The action of t on M is a linear map $T : M \to M$.

(4) Explain the relationship between $\mathrm{Ann}(M)$ and the minimal polynomial of T.
(5) Explain the relationship between $\mathrm{End}(M)$ and the ***centralizer*** $\{B \in \mathrm{Hom}_k(M, M) : TB = BT\}$.

The matrix T is called ***regular*** if the conditions of part 3 hold.

Problem 5. This is a corrected version of Problem 8 from the previous problem set. The goal is to show that, if R is a UFD in which every prime ideal is maximal, then R is a PID. I know many students found proofs of this despite the bad hint, so I will accept any proof of this statement for this problem. If you wish, you may assume R is Noetherian. Here is a sketch of what I now think is the clearest route:

(1) For a and $b \in R$ show that, if a and b are relatively prime, then aR and bR are comaximal.
(2) Show that, for any finite list of elements $a_1, a_2, \dots, a_n$ of R, the ideal $\langle a_1, a_2, \dots, a_n \rangle$ is the principal ideal $\mathrm{GCD}(a_1, a_2, \dots, a_n)R$.
(3) Show that, for any subset A of R, the ideal generated by A is the principal ideal $\mathrm{GCD}(A)R$.
(4) Show that R is a PID.

I'll grade this out of 40 points, assigning 10 for each of the steps below if you follow this outline, or figuring out something else that seems fair if you want to take a different route.