PROBLEM SET 11: DUE DECEMBER 8

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 11. **Both questions will be about exterior algebra.**

Problem 2. Please write up proofs of **three** of problems **20.2, 20.5, 21.4, 21.5, 21.6, 21.7**.

Problem 3. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$ and consider the map $\left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{smallmatrix}\right] : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Compute the matrix of $\bigwedge^2 \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{smallmatrix}\right]$ in the basis $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_2 \wedge e_3$ for $\bigwedge^2 \mathbb{R}^3$.

Problem 4. Recall that the rank of a linear map $\phi : V \to W$ is the dimension of $\phi(V)$. Show that $\bigwedge^k \phi = 0$ if and only if the rank of ϕ is $< k$. Please define the rank as the dimension of the image.

Problem 5. (**The rational root theorem.**) Let R be a UFD and let $K = \mathrm{Frac}(R)$. Let $f(x) = f_n x^n + \cdots + f_1 x + f_0 \in R[x]$ and suppose that $f(a/b) = 0$ for $a/b \in K$, with $\mathrm{GCD}(a, b) = 1$. Show that a divides f_0 and b divides f_n.

Problem 6. In this problem, we use the rational root theorem (Problem 5) to provide another perspective on Gauss's lemma. Recall that for $R \subseteq S$ and $\theta \in S$; we defined θ to be integral over R if θ is a zero of a **monic** polynomial in $R[x]$. (See Problem Set 5, Problem 7.)

Let R be a domain, let $K = \mathrm{Frac}(R)$ and let $f(x) = x^m + f_{m-1}x^{m-1} + \cdots + f_0$ be a monic polynomial in $R[x]$. Let L be an algebraic closure of K and let $f(x)$ factor as $\prod(x - \theta_i)$ in L.

1. Let $g(x) = x^n + g_{n-1}x^{n-1} + \cdots + g_1 x + g_0$ be a monic polynomial in $L[x]$ dividing $f(x)$. Show that the coefficients g_i are integral over R.
2. Now suppose that R is a UFD, and suppose that $f(x) = g(x)h(x)$ with g and h monic polynomials in $K[x]$. Show that $g(x)$ and $h(x)$ are in $R[x]$ (Hint: See Problem 5.)

This is a large fraction of Gauss's lemma, and working a bit harder this can be turned into a full proof of Gauss's lemma using the concept of integral elements.

Problem 7. This problem presents the basics of symmetric bilinear forms. Let k be a field of **characteristic not equal to** 2 and let V be a vector space over k. Let $\langle\ ,\ \rangle : V \times V \to k$ be a symmetric bilinear form, meaning that $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$, $\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$, $\langle cv, w\rangle = \langle v, cw\rangle = c\langle v, w\rangle$ and $\langle v, w\rangle = \langle w, v\rangle$.

1. Suppose that $\langle v, v\rangle = 0$ for all v in V. Show that $\langle v, w\rangle = 0$ for all v and $w \in V$.
2. Let $\langle\ ,\ \rangle : V \times V \to k$ be a symmetric bilinear form and let $v \in V$ with $\langle v, v\rangle \neq 0$. Set $v^\perp = \{w \in V : \langle v, w\rangle = 0\}$. Show that $V = kv \oplus v^\perp$.
3. Let $\dim V < \infty$. Show that V has a basis $v_1, \ldots, v_n$ such that $\langle v_i, v_j\rangle = 0$ for $i \neq j$.

Given two bilinear forms $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ on V, we say that they are isomorphic if there is an automorphism $\phi : V \to V$ such that $\langle v_1, v_2\rangle_1 = \langle \phi(v_1), \phi(v_2)\rangle_2$.

4. Consider the three symmetric bilinear forms
$$\begin{aligned} \langle (x_1, x_2), (y_1, y_2)\rangle_{++} &= x_1y_1 + x_2y_2 \\ \langle (x_1, x_2), (y_1, y_2)\rangle_{+-} &= x_1y_1 - x_2y_2 \\ \langle (x_1, x_2), (y_1, y_2)\rangle_{--} &= -x_1y_1 - x_2y_2 \end{aligned}$$
on $\mathbb{R}^2$. Show that these are not isomorphic.
5. Now let k be the field with 3 elements. Define bilinear forms $\langle\ ,\ \rangle_{++}$, $\langle\ ,\ \rangle_{+-}$ and $\langle\ ,\ \rangle_{--}$ on k^2 as above. Two of them are isomorphic to each other; which ones?