PROBLEM SET TWO: DUE SEPTEMBER 22

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 2.

Problem 2. Write up complete solutions to **two** of the following problems from class:
6.4, 7.3, 7.4, 7.5, 8.2, 8.3

Problem 3. We recall that an element R of a ring e is called idempotent if $e^2 = e$. A collection of idempotents $(e_1, e_2, \ldots, e_n)$ is called ***orthogonal*** if $e_i e_j = 0$ for $i \neq j$. A collection of orthogonal idempotents is called ***complete*** if $\sum e_i = 1$.

(1) Let R be a ring and let $(e_1, e_2, \ldots, e_n)$ be a complete collection of orthogonal idempotents. Show that, as an abelian group, $R = \bigoplus_{i,j} e_i R e_j$.
(2) If $f \in e_i R e_j$ and g is in $e_k R e_\ell$, what can you say about where fg lies in the decomposition $R = \bigoplus_{i,j} e_i R e_j$? Can you use this to write elements of R as matrices in a useful way?
(3) Suppose that e_1, e_2, …, e_n are *commuting* orthogonal idempotents. Consider the 2^n products of the form $f_1 f_2 \cdots f_n$ where each f_i is either e_i or $1 - e_i$. Show that these 2^n products are a complete collection of orthogonal idempotents.
(4) Suppose that $(e_1, e_2, \ldots, e_n)$ is a complete collection of orthogonal idempotents, all of which are *central* in the ring R. (Meaning that $re_i = e_i r$ for all $r \in R$.) Show that R is isomorphic, as a ring, to $\prod_{i=1}^{n} e_i R e_i$.

Problem 4. Let R be a commutative ring. R is called ***local*** if R has precisely one maximal ideal. Show that a ring A is local if and only if the set of non-units in A forms an ideal of A.

Problem 5. This problem displays some applications of the Chinese Remainder Theorem over $\mathbb{Z}$.

(1) Let n be a positive integer with prime factorization $n = \prod p_j^{e_j}$. Give a formula for the number of ordered pairs $(a, b) \in \{0, 1, 2, \ldots, n-1\}^2$ such that $\mathrm{GCD}(a, b, n) = 1$.
(2) An integer n is called squarefree if it is not divisible by k^2 for any $k > 1$. Show that there is some integer N such that N, $N+1$, …, $N+2019$ are all **not** squarefree.
(3) Find an integer c, which is even and not divisible by 5, such that $5^k + c$ is **not** prime for any positive integer k. Feel free to use a computer algebra system to help.

Problem 6. Let R be a ring, and let A, B and C be R-modules. Suppose that we have a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, meaning that α is injective, β is surjective and $\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$.

(1) Suppose that we have a map $\rho : B \to A$, satisfying $\rho \circ \alpha = \mathrm{Id}_A$. Show that $B \cong A \oplus C$.
(2) Suppose that we have a map $\sigma : C \to B$, satisfying $\beta \circ \sigma = \mathrm{Id}_C$. Show that $B \cong A \oplus C$.

Problem 7. Let R be the ring of integer quaternions: R is a free $\mathbb{Z}$-module with basis 1, i, j, k, and multiplication $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Let p be an odd positive prime integer.

(1) Show that there are integers u and v with $u^2 + v^2 + 1 \equiv 0 \bmod p$. (Hint: Pigeonhole principle.)
(2) Show that there is a well-defined map of rings $R/pR \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Z}/p\mathbb{Z})$ with $i \mapsto \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $j \mapsto \left[\begin{smallmatrix} u & v \\ v & -u \end{smallmatrix}\right]$.
(3) Show that the map in (2) is an isomorphism.
(4) Show that R has a left ideal J with $|R/J| = p^2$.