PROBLEM SET THREE: DUE SEPTEMBER 29

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 3.

Problem 2. Write up complete solutions to **two** of the following problems from class:

9.1, 9.6, 10.1, 10.5, 10.6, 11.1, 11.2, 11.5

Problem 3. Let R be a finite ring, and let $|R|$ factor as $\prod p^{a_p}$. Show that there are rings R_p, for the various primes p dividing N, such that $|R_p| = p^{a_p}$ and $R \cong \prod R_p$.

Problem 4. Describe all simple modules for the following rings:

(1) $\mathbb{Z}/15\mathbb{Z}$.
(2) The ring $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ of 2×2 matrices with entries in $\mathbb{R}$.
(3) The subring $\{\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]\}$ of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

Problem 5. For the given ring R and module M, give a composition series of M:

(1) ~~$R = \mathbb{Z}$, $M = \mathbb{Z}/4\mathbb{Z}$~~. Done in class.
(2) ~~$R = \mathbb{Z}$, $M = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.~~ Done in class.
(3) $R = \mathbb{R}[x, y]$, $M = R/\langle x^2, xy, y^2\rangle$.
(4) $R = \mathbb{R}[S_3]$, $M = \mathbb{R}^3$, where the symmetric group S_3 permutes the coordinates in $\mathbb{R}^3$.
(5) $R = (\mathbb{Z}/3\mathbb{Z})[S_3]$, $M = (\mathbb{Z}/3\mathbb{Z})^3$, where the symmetric group S_3 permutes the coordinates in $(\mathbb{Z}/3\mathbb{Z})^3$.

Problem 6. (**Schur's lemma, continued**) Let k be a field and let R be a ring which contains k in its center, meaning that, for $a \in k$ and $r \in R$, we have $ar = ra$. Let S be a **simple** R-module which is finite dimensional as a k-vector space. Let $\phi : S \to S$ be an endomorphism of S.

(1) Let $f(x)$ be the minimal polynomial of the k-linear map ϕ. Show that $f(x)$ is irreducible in $k[x]$.
(2) Suppose that k is algebraically closed. Show that there is a scalar $c \in k$ such that $\phi(v) = cv$ for all $v \in S$.

Problem 7. Let R be a ring, let M be an R-module and let E be the endomorphism ring $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$. Show that M decomposes as $M_1 \oplus M_2$, with M_1 and M_2 both nonzero, if and only if E contains an idempotent other than 0 and 1.

Problem 8. Let R be a ring, let M and N be R-modules and let $\alpha : M \to N$ and $\beta : N \to M$ be module homormorphisms such that $\beta \circ \alpha = \mathrm{Id}_M$. Show that there is a submodule K of N such that $N \cong M \oplus K$.

Problem 9. Postponed to problem set 4. ~~Let R be a ring and let M be an R-module of finite length. Let ϕ be an endomorphism of M. Define $I_n = \mathrm{Image}(\phi^n)$ and define $K_n = \mathrm{Ker}(\phi^n)$.~~

(1) ~~Show that there is a positive integer N such that $I_N = I_{N+1} = I_{N+2} = \cdots$ and $K_N = K_{N+1} = K_{N+2} = \cdots$~~
(2) ~~For the N above, show that $M = I_N \oplus K_N$.~~
(3) ~~Suppose that M cannot be written as a direct sum $M_1 \oplus M_2$, with M_1 and M_2 both nonzero modules. Show that every endomorphism of M is either nilpotent or invertible.~~