PROBLEM SET FOUR: DUE OCTOBER 6

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 4. Note that the second question addresses Noetherian rings.

Problem 2. Write up complete solutions to **one** of the following problems from class: 11.2, 11.5

In class, we gave a fairly sophisticated proof of Worksheet Problem 11.1. I don't want you to think it needs to be this sophisticated, so the next problem walks you through a direct attack.

Problem 3. We recall the set up: Let R be a ring, let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of R-modules and let $B_1 \subset B_2 \subset B$ be submodules such that B_2/B_1 is simple. Set $A_j = \alpha^{-1}(B_j)$ and $C_j = \beta(B_j)$.

(1) Show that the following are equivalent: $A_2 = A_1$ and $\alpha(A) \cap B_2 = \alpha(A) \cap B_1$.
(2) Show that the following are equivalent: $C_2 = C_1$ and $\alpha(A) + B_2 = \alpha(A) + B_1$.

We want to show that exactly one of these conditions holds.

(3) Suppose that $\alpha(A) \cap B_2 = \alpha(A) \cap B_1$. Take x_2 in B_2 and not in B_1. Show that $x_2 \in \alpha(A) + B_2$ and $x_2 \notin \alpha(A) + B_1$.
(4) Let x_2 be in B_2 and not B_1, and let y_2 be any other element of B_2. Show that there is $r \in R$ and $x_1 \in B_1$ with $y_2 = rx_2 + x_1$.
(5) Supposes that $\alpha(A) \cap B_2 \neq \alpha(A) \cap B_1$. Choose x_2 in $\alpha(A) \cap B_2$ and not in $\alpha(A) \cap B_1$. Take any element of $y_2 + \alpha(a)$ of $\alpha(A) + B_2$ and show that it is in $\alpha(A) + B_1$.

Problem 4. Let E be a ring and let x be a nilpotent element of E. Show that $1 - x$ is a unit.

Problem 5. Let R be a ring and let M be an R-module of finite length.

(1) Let $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ be a chain of R-submodules of M. Show that there is an index N such that $A_N = A_{N+1} = A_{N+2} = \cdots$.
(2) Let $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$ be a chain of R-submodules of M. Show that there is an index N such that $B_N = B_{N+1} = B_{N+2} = \cdots$.

Problem 6. Let R be a ring and let M be an R-module of finite length. Let ϕ be an endomorphism of M. Define $I_n = \text{Image}(\phi^n)$ and define $K_n = \text{Ker}(\phi^n)$.

(1) Show that there is an index N such that $I_N = I_{N+1} = \cdots$ and $K_N = K_{N+1} = \cdots$.
(2) For the N above, show that $M = I_N \oplus K_N$.
(3) Suppose that M cannot be written as a direct sum $M_1 \oplus M_2$, with M_1 and M_2 both nonzero modules. Show that every endomorphism of M is either nilpotent or invertible.

Problem 7. This problem works through a common example of a kind of ring we will study: Let $f(x) = x^n + f_{n-1}x^{n-1} + \cdots + f_1 x + f_0$ be a monic irreducible polynomial with coefficients in $\mathbb{Z}$. Let θ be a root of $f(x)$ in $\mathbb{C}$ and let $\mathbb{Z}[\theta]$ be the subring of $\mathbb{C}$ generated by θ.

(1) Show that $\mathbb{Z}[\theta]$ is a free $\mathbb{Z}$-module with basis $1, \theta, \ldots, \theta^{n-1}$. In other words, show that every element of $\mathbb{Z}[\theta]$ can be written as $\sum_{j=0}^{n-1} a_j \theta^j$ for $a_j \in \mathbb{Z}$ in precisely one way.
(2) Show that $\mathbb{Z}[x]/f(x)\mathbb{Z}[x] \cong \mathbb{Z}[\theta]$.
(3) This problem studies the ring $\mathbb{Z}[i]$ where i is a square root of -1. Let p be a prime integer. Show that the ideal $p\mathbb{Z}[i]$ is prime if and only if there is **not** an $x \in \mathbb{Z}$ with $x^2 \equiv -1 \bmod p$.
(4) Let $R_3 = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ and $R_7 = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$. Show that $R_3/2R_3$ is a field with four elements and that $R_7/2R_7 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.