PROBLEM SET FIVE: DUE OCTOBER 13

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 5.

Problem 2. From Worksheet 12, please write up an implication of each of the following forms:

$$x(a) \implies x(b),\ x(b) \implies x(c),\ 1(y) \implies 2(y),\ 2(y) \implies 3(y) \text{ and } 3(y) \implies 1(y)$$

for some $x \in \{1, 2, 3\}$ and some $y \in \{a, b, c\}$.

Problem 3. (1) Show that any field k is Noetherian.
(2) Show that $\mathbb{Z}$ is Noetherian.
(3) Let R be a ring which contains a field k and is finite dimensional as a k vector space. Show that R is noetherian.

Problem 4. Show that the following rings are **not** Noetherian:

(1) The polynomial ring $k[x_1, x_2, \ldots]$ in infinitely many variables.
(2) The union $\bigcup_{n=0}^{\infty} k[x^{1/2^n}]$. In other words, we have inclusions $k[x] \subset k[x^{1/2}] \subset k[x^{1/4}] \subset \cdots$, and we take the union of all of them.
(3) The subring of $k[x, y]$ generated by all monomials of the form $x^j y$.

Problem 5. Let R be a Noetherian ring and let I be a two-sided ideal. Show that R/I is Noetherian.

Problem 6. In this problem, we prove Hilbert's basis theorem: If R is a Noetherian commutative ring, then $R[x]$ is Noetherian. Let I be an ideal of $R[t]$. We will show that I is finitely generated. Define $I_d := \{f \in R : \text{there is an element of } I \text{ of the form } fx^d + f_{d-1}x^{d-1} + \cdots + f_1 x + f_0\}$.

(1) Show that I_d is an ideal of R and that $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$.
(2) Show that there is an index r such that $I_r = I_{r+1} = I_{r+2} = \cdots$. Show that I_r is finitely generated as an R-module.
(3) Let M be the set of polynomials in I with degree $\leq r$. Show that M is finitely generated as an R-module.

Let $f_1, f_2, \ldots, f_m$ generate I_r as an R-module and choose elements g_j of I of the form $g_j = f_j x^r + (\text{lower order terms})$. Let $h_1, h_2, \ldots, h_n$ generate M as an R-module.

(4) Show that $g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n$ generate I as an $R[x]$ module.

Problem 7. Let $R \subseteq S$ be commutative rings. For $\theta \in S$, we write $R[\theta]$ for the set of elements of S of the form $r_n\theta^n + r_{n-1}\theta^{n-1} + \cdots + r_1\theta + r_0$.

(1) Show that $R[\theta]$ is finitely generated as an R-module if and only if there is a monic polynomial $f(x) = x^n + f_{n-1}x^{n-1} + \cdots + f_1 x + f_0$ with coefficients f_j in R, such that $f(\theta) = 0$.

When these equivalent conditions are satisfied, we say that θ is ***integral over*** R.

(2) Let α and β in S be integral over R. Show that $R[\alpha, \beta]$ is finitely generated as an R-module.
(3) Let α and β in S be integral over R and let ϕ be any element of $R[\alpha, \beta]$. Show that ϕ is integral over R. You may assume that R is Noetherian, although you don't need to. [1]

So the set of elements of S which are integral over R forms a subring of S.

[1] A hint for how to avoid the hypothesis that R is Noetherian: Use the Cayley-Hamilton theorem.