Problem Set Six: Due October 27

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 6.

Problem 2. Write up complete solutions to ~~**three**~~ **two** of the following problems from class:
13.5, 13.7, 13.8, 14.5, ~~14.8~~, 16.5

Problem 3. In the ring $\mathbb{Z}[x]$, is 15 a unit, irreducible or a composite? What about in the ring $\mathbb{Q}[x]$?

Problem 4.
(1) Compute the GCD of 2021 and 215 using the Euclidean algorithm.
(2) Find integers x and y such that $2021x + 215y = \mathrm{GCD}(2021, 215)$.

Problem 5. Let k be a field and let $b_1(t)$, $b_2(t)$, $\dots$, $b_r(t)$ be pairwise relatively prime polynomials in $k[t]$. Set $g(t) = b_1(t)b_2(t)\cdots b_r(t)$.

(1) Show that the polynomials $g(t)/b_j(t)$ generate $k[t]/g(t)k[t]$ as a $k[t]$-module.
(2) Let $f(t) \in k[t]$. Show that there are polynomials $a_1(t), \dots, a_r(t)$ and $c(t)$ in $k[t]$ such that

$$\frac{f(t)}{g(t)} = \sum_{j=1}^{r} \frac{a_j(t)}{b_j(t)} + c(t).$$

Now you know why integration by partial fractions works!

Problem 6. Let R be a PID and let x be a nonzero element of R. Let $M = R/xR$ and let y be another nonzero element of R.

(1) Show that $yM \cong R/\frac{x}{\mathrm{GCD}(x,y)}R$ as R-modules.
(2) Give similar descriptions for the R-modules M/yM and $M[y] := \{m \in M : ym = 0\}$.
(3) Show that M has finite length as an R-module, and describe $\ell(M)$ in terms of a prime factorization of x.

Problem 7. Let R be a PID.

(1) Let x and $y \in R$ with $\mathrm{GCD}(x, y) = g$. Show that there is a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with entries in R such that $ad - bc = 1$ and $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} g \\ 0 \end{smallmatrix}\right]$.
(2) Let $x_1, x_2, \dots, x_n \in R$ with $\mathrm{GCD}(x_1, x_2, \dots, x_n) = g$. Show that there is an $n \times n$ matrix A with entries in R such that $\det A = 1$ and $A \left[\begin{smallmatrix} x_1 & x_2 & \cdots & x_n \end{smallmatrix}\right]^T = \left[\begin{smallmatrix} g & 0 & \cdots & 0 \end{smallmatrix}\right]^T$.
(3) Let x and $y \in R$. Show that there are invertible 2×2 matrices U and V with

$$U \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} V = \begin{bmatrix} \mathrm{GCD}(x,y) & 0 \\ 0 & \mathrm{LCM}(x,y) \end{bmatrix}.$$

Problem 8. Let R be a commutative ring. Let A and B be $m \times n$ matrices with entries in R and suppose that there exist invertible matrices U and V, of sizes $m \times m$ and $n \times n$ such that $B = UAV$.

(1) Show that the ideal generated by all the entries of A is the same as the ideal generated by all the entries of B.
(2) Take $R = \mathbb{Q}[x, y]$, $A = \left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & xy \end{smallmatrix}\right]$. Show that there do **not** exist invertible matrices U and V with $B = UAV$.
(3) With R, A and B as above, are the modules R^2/AR^2 and R/BR^2 isomorphic or not?

Problem 9. Let R be an integral domain and let p_1, p_2, $\dots$, p_k be elements of R such that p_1R, p_2R, $\dots$, p_kR are distinct **maximal** ideals. Let I be an ideal containing $p_1p_2\cdots p_k$. Show that I is $p_{i_1}p_{i_2}\cdots p_{i_s}R$ for some subset $\{i_1, i_2, \dots, i_s\}$ of $\{1, 2, \dots, k\}$.