PROBLEM SET 7: DUE NOVEMBER 3

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 7.

Problem 2. Please write up proofs of the following class problems: **17.5**, and either **18.1** or **18.4**.

Problem 3.
(1) Let $A = \left[\begin{smallmatrix} 2 & 2 \\ 2 & 6 \end{smallmatrix}\right]$. Write A in Smith normal form.
(2) Let $B = \left[\begin{smallmatrix} 2 & 4 & 10 \\ 1 & 3 & 7 \\ 1 & 1 & 15 \end{smallmatrix}\right]$. If we were to write B in Smith Normal form as UDV, what would D be? (You need not find U and V.)

Problem 4.
(1) Use the Euclidean algorithm to find polynomials $f(t)$ and $g(t)$ in $\mathbb{Q}[t]$ with
$$f(t)(3t^2 - 3t - 1) + g(t)(t^3 - 2) = 1.$$
(2) Find rational numbers a, b, c such that
$$(3\sqrt[3]{4} - 3\sqrt[3]{2} - 1)^{-1} = a\sqrt[3]{4} + b\sqrt[3]{2} + c.$$
Now you can rationalize denominators for algebraic numbers of degree greater than 2!

Problem 5. Let R be an integral domain and let I be a nonzero ideal of R.

(1) Draw arrows indicating which implications exist between the following concepts. You need not provide proofs or counterexamples:

I is prime	I is maximal
I is of the form (f) for f irreducible	I is of the form (f) for f prime

(2) How would your answers change if we assume that R is a UFD?
(3) How would your answers change if we assume that R is a PID?

Problem 6. Let R be a commutative ring and let A be an $n \times n$ matrix with entries in R. The point of this problem is to prove that the following are equivalent:

(a) $\det A$ is a unit of R.
(b) There is an $n \times n$ matrix B with $AB = \mathrm{Id}_n$.
(c) There is an $n \times n$ matrix C with $CA = \mathrm{Id}_n$.

For ease of grading, break up your proof as follows:

(1) Show that (b) or (c) implies (a).
(2) Show that (a) implies (b) and (c). Hint: Look up the adjugate matrix.
(3) Show that, if (b) and (c) hold, then $B = C$.

Problem 7. The following problem gives a criterion for a ring to be a PID which is similar to, but more general, than being Euclidean. Let R be an integral domain and let $N(\ )$ be a positive norm on R. Let P be a subset of R such that, for all a and $b \in R$ with $b \neq 0$, there are $p \in P$ and q and $r \in R$ such that $pa = qb + r$ and $N(r) < N(b)$.

Let J be a nonzero ideal of R and let b be a nonzero element of J of minimal norm.

(1) Show that, for every $a \in J$, there is some $p \in P$ with $pa \in bR$.
(2) Now suppose that R is Noetherian. Show that there are distinct elements $p_1, p_2, \ldots, p_k$ of R with $(\prod p_i)J \subseteq bR$.
(3) Suppose that R is Noetherian and that pR is maximal for all $p \in P$. Show that J is principal. You may use Problem 9 from Problem Set 6 even if you haven't solved it.