Problem Set 8: Due November 10

Problem 1. Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 8.

Problem 2. Please write up solutions to **two** of the following problems: **19.1**, **19.3**, **19.5**.

Problem 3. Set

$$M = \begin{bmatrix} 2 & 4 & 10 \\ 1 & 3 & 7 \\ 1 & 1 & 15 \end{bmatrix}.$$

Let G be the abelian group $\mathbb{Z}^3/M\mathbb{Z}^3$. Write G as a product of cyclic groups of prime power order.

Problem 4. Let A be a finite abelian group with n elements. Suppose that, for each prime p dividing n, there are only p solutions to $pa = 0$ in A. Show that $A \cong \mathbb{Z}/n\mathbb{Z}$.

Problem 5. Let R be a PID and let A be an $n \times n$ matrix with entries in R, such that $\det A \neq 0$. Let $M = R^n/AR^n$.

(1) In the case that $R = \mathbb{Z}$, show that $\#(M) = |\det A|$.
(2) Show that M has finite length.
(3) Let $d_1, d_2, \ldots, d_n$ be the invariant factors of A. Let p be a prime element of R. Describe the number of times the simple module R/pR will occur as a quotient in the Jordan-Holder filtration of M, in terms of the factorizations of the d_i.
(4) Show that M and $R/(\det A)R$ have the same simple quotients in their Jordan-Holder filtration.[1]

Problem 6. Let A be an $n \times n$ matrix with entries in $\mathbb{Z}/p\mathbb{Z}$ and determinant 1. Show that there is an $n \times n$ matrix B with entries in $\mathbb{Z}$ and determinant 1 such that $A \equiv B \bmod p$. (Hint: Think about elementary matrices.)

Problem 7. Let p be a prime integer and let A be an abelian group. Describe all possible groups A if we have a short exact sequence of the following forms:

(1) $0 \to \mathbb{Z}/p\mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0$.
(2) $0 \to \mathbb{Z}/p\mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \to 0$.
(3) $0 \to \mathbb{Z}/p^2\mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \to 0$.

Problem 8. (1) Let A and B be matrices with entries in $\mathbb{Z}$, of sizes $r \times s$ and $s \times t$ respectively, and let $C = AB$; we write A_{ij}, B_{jk} and C_{ik} for the entries of these matrices. **Prove or disprove:** $\mathrm{GCD}(C_{ik}) = \mathrm{GCD}(A_{ij})\,\mathrm{GCD}(B_{jk})$. (The left hand side is the GCD of all entries of C, and similarly for A and B.)
(2) Let $\sum_{i=0}^{m} a_i x^i$ and $\sum_{j=0}^{n} b_j x^j$ be polynomials with coefficients in $\mathbb{Z}$ and let $\sum_{k=0}^{m+n} c_k x^k = \left(\sum a_i x^i\right)\left(\sum b_j x^j\right)$. **Prove or disprove:** $\mathrm{GCD}(c_k) = \mathrm{GCD}(a_i)\,\mathrm{GCD}(b_j)$. (The left hand side is the GCD of all coefficients of $c(x)$, and similarly for $a(x)$ and $b(x)$.)

[1] This is actually true whenever R is an integral domain such that R/aR is finite length for all nonzero a. Perhaps I'll find a nice enough proof to assign in a later problem set.