# PROBLEM SET 9: DUE NOVEMBER 17

**Problem 1.** Remember to go to plan an hour to go to Gradescope and do Practice QR Exam 9.

**Problem 2.** (This problem replaces the usual worksheet write ups:) Let $k$ be a field.

(1) Let $f(x) = x^d + f_{d-1}x^{d-1} + \cdots + f_1 x + f_0$ be a monic polynomial with coefficients in $x$. Show that $x^{d-1}$, $x^{d-2}$, $\ldots$, $x$, $1$ is a basis of $k[x]/f(x)k[x]$ and write down the matrix for multiplication by $x$ in that basis.
(2) Let $\lambda \in k$. Show that $(x-\lambda)^{n-1}$, $(x-\lambda)^{n-2}$, $\ldots$, $x-\lambda$, $1$ is a basis of $k[x]/(x-\lambda)^n k[x]$ and write down the matrix for multiplication by $x$ in that basis.

**Problem 3.** Let $R$ be a PID; let $M$ be a submodule of $R^{\oplus n}$. Show that $M \cong R^{\oplus m}$ for some $m \leq n$.

**Problem 4.** Let $k$ be an algebraically closed field. Let $X$ be an $n \times n$ matrix with entries in $k$, and let the Jordan blocks of $X$ be $J_{n_1}(\lambda_1)$, $J_{n_2}(\lambda_2)$, $\ldots$, $J_{n_r}(\lambda_r)$. Express the following quantities in terms of the $\lambda_j$ and $n_j$ and prove your answer to be correct.

(1) The characteristic polynomial of $X$.
(2) The minimal polynomial of $X$.
(3) The dimension of $\operatorname{Ker}(X - \lambda \mathrm{Id})$.

**Problem 5.** Consider the nilpotent matrix whose powers are shown below:

$$X = \begin{bmatrix} -1 & -7 & -2 & -2 \\ 0 & 3 & 1 & 1 \\ 0 & -6 & -2 & -2 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad X^2 = \begin{bmatrix} -1 & -4 & -1 & -1 \\ 1 & 4 & 1 & 1 \\ -2 & -8 & -2 & -2 \\ -1 & -4 & -1 & -1 \end{bmatrix} \quad X^3 = 0$$

$$\operatorname{rank}(X) = 2 \qquad \operatorname{rank}(X^2) = 1 \qquad \operatorname{rank}(X^3) = 0$$

Find the Jordan normal form of $X$.

**Problem 6.** Let $R$ be a PID. The first part of this problem is Problem 8, part 2, with a hint, so you may simply write "I solved this last time" if you did so.

(1) Let $A(x) = \sum a_i x^i$ and $B(x) = \sum b_j x^j$ be polynomials in $R[x]$ and let $C(x) = \sum c_k x^k$ be their product. Show that $\operatorname{GCD}(c_j) = \operatorname{GCD}(a_i)\operatorname{GCD}(b_k)$. Hint: Show that, if a prime $p$ divides all of the $c_k$, then either $p$ divides all of the $a_i$ or else $p$ divides all of the $b_j$.
(2) Let $K$ be the field of fractions of $R$. Let $C(x)$ be a polynomial in $R[x]$ which factors as $A(x)B(x)$ for $A$ and $B$ in $K[x]$. Show that there is a nonzero scalar $\phi$ in $K$ such that $\phi A(x) \in R[x]$ and $\phi^{-1}B(x) \in R[x]$.
(3) (**Eisenstein's criterion**) Let $p$ be a prime and let $f(x) = f_n x^n + \cdots + f_1 x + f_0$ be a polynomial with coefficients in $R[x]$. Suppose that $p$ does not divide $f_n$, that $p$ does divide $f_{n-1}$, $f_{n-2}$, $\ldots$, $f_1$, $f_0$ and $p^2$ does not divide $f_0$. Show that $f(x)$ is irreducible.

**Problem 7.** We return to Problem 7 of Problem Set 7 to show how it can be used. Let $R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$. Define a norm on $R$ by $N\left(\frac{u+v\sqrt{-19}}{2}\right) = \frac{u^2+19v^2}{4}$.

(1) Show that, for all $a$ and $b \in R$, with $b \neq 0$, either there exist $q$ and $r$ with $a = bq + r$ and $N(r) < N(b)$ or there exist $q$ and $r$ with $2a = bq + r$ and $N(r) < N(b)$. Hint: Adapt the criterion from Worksheet Problem 16.3.
(2) Show that $(2)$ is a prime ideal of $R$.
(3) Conclude that $R$ is a PID.

**Problem 8.** ~~Let $R$ be a Noetherian UFD in which all prime ideals are maximal. Using the criterion of Problem Set 6, Problem 9, show that $R$ is a PID.~~ This problem is removed because the hint was bad. The main result is true: If $R$ is a UFD where all prime ideals are maximal then $R$ is a PID. (One doesn't even need Noetherian here.) The hint I provided was a bad one. I saw at least two students in office hours who found proofs ignoring the hint; on a future problem set, I'll bring this back with a better hint.