BILINEAR FORMS

Suppose k is a field and V is a k-vector space.

Definition. A *k-bilinear form on V* is a bilinear pairing $B : V \times V \to k$. A k-bilinear form B is said to be

- ***symmetric*** provided that $B(x, y) = B(y, x)$ for all x and $y \in V$,
- ***alternating*** provided that $B(u, u) = 0$ for all $u \in V$, and
- ***skew-symmetric*** (or *anti-symmetric*) provided that $B(s, t) = -B(t, s)$ for all s and $t \in V$.

(183) Show that every alternating form is skew symmetric. Hint for this problem and the next two: Think about $B(v + w, v + w)$.

(184) Show that, if the characteristic of k is not 2, then every skew-symmetric form is alternating.

(185) Show that, if the characteristic of k is not 2 and B is a symmetric bilinear form with $B(v, v) = 0$ for all $v \in V$, then $B(v, w) = 0$ for all v and $w \in V$.

We now restrict our attention to the finite dimensional case:

(186) Let $v_1, v_2, \ldots, v_n$ be a basis of V and let G be the $n \times n$ matrix $G_{ij} = B(v_i, v_j)$. We call G the ***Gram matrix***.[1]
   (a) In the basis $v_1, \ldots, v_n$, verify the formula $B(\vec{x}, \vec{y}) = \vec{x}^T G \vec{y}$.
   (b) Under what conditions on G will B be symmetric?
   (c) Under what conditions on G will B be alternating?
   (d) Under what conditions on G will B be skew-symmetric?

(187) Let $w_1, w_2, \ldots, w_n$ be a second basis of V, with $w_j = \sum S_{ij} v_i$. Let H be the Gram matrix $B(w_i, w_j)$. Give a formula for H in terms of S and G.

A bilinear form B on V is called ***nondegenerate*** if, for all $v \in V$, there is some $w \in V$ with $B(v, w) \neq 0$.

(188) Let V be a finite dimensional vector space. Show that B is nondegenerate if and only if the Gram matrix of B is invertible.

(189) Let V be a finite dimensional[2] vector space, let B be a bilinear form on V and let L be a subspace of V such that the restriction of B to L is nondegenerate. Define $L^\perp = \{v \in V : B(u, v) = 0 \ \forall u \in L\}$. Show that $V = L \oplus L^\perp$.

Note: In class, I thought that Problem 189 would break if B were not symmetric. In fact, the problem is right as written. However, it is only in the symmetric case that we will have $B(u, v) = 0$ for $u \in L^\perp$ and $v \in L$. In general, let $L^\perp = \{v \in V : B(u, v) = 0 \ \forall u \in L\}$ and let ${}^\perp L = \{v \in V : B(v, u) = 0 \ \forall u \in L\}$. Then under the hypotheses of Problem 189, it is true both that $V = L \oplus L^\perp$ and that $V = L \oplus {}^\perp L$. However, it is only in the symmetric and skew symmetric cases that we will have $L^\perp = {}^\perp L$.

[1] The term Gram matrix is generally used in the context of applied linear algebra, such as computer graphics and control theory. In that context, the vector space V is simply $\mathbb{R}^n$ and B is simply dot product, but v_i is some basis of $\mathbb{R}^n$ which is not orthonormal. The Gram matrix encodes the "skewness" of our basis.

[2] Without finite dimensionality, this is not true. Let V be a vector space with basis $e_1, e_2, e_3, \ldots$ and consider the standard bilinear form $B\left(\sum a_i e_i, \sum b_i e_i\right) = \sum a_i b_i$. Let L be the subspace spanned by $e_i - e_j$. Then $L^\perp$ is 0 because, if $\sum a_k e_k$ is perpendicular to all $e_i - e_j$ then $a_i = a_j$ for all i, j. But V only allows finite sums, so the only such element are 0.