Comaximal Ideals

Vocabulary: Product of ideals, comaximal ideals

We now introduce the notion of comaximal ideals. As we will see, ideals being comaximal is something like integers being relatively prime.

Definition. Suppose R is a commutative ring. Ideals A, B of R are said to be ***comaximal*** provided that $A + B = R$.

(33) Show that A and B are comaximal if and only if $1 \in A + B$.
(34) If $\mathfrak{m}$ is maximal and I is an ideal, show that either $\mathfrak{m}$ and I are comaximal, or else $I \subseteq \mathfrak{m}$.
(35) Let R be a commutative ring and let A and B be ideals. Show that the map $R \to R/A \times R/B$, sending r to the ordered pair $(r \bmod A, r \bmod B)$, is surjective if and only if A and B are comaximal.[1]

Definition. Suppose R is a ring. The ***product*** of ideals A and B in R is the ideal, denoted AB, consisting of all finite sums $\sum a_i b_i$ with $(a_i, b_i) \in A \times B$. The product of any finite number of ideals is defined similarly.

(36) Suppose that A and B are comaximal ideals in a commutative ring R. Show that $A \cap B = AB$.
(37) Suppose that R is a nonzero commutative ring. Suppose I_1, I_2, $I_3, \ldots, I_k$ are ideals in R that are pairwise comaximal. Show that the ideals I_1 and $I_2 I_3 \cdots I_k$ are comaximal.

We now show that comaximal is a stronger condition than relatively prime, and is the same in $\mathbb{Z}$.

(38) Let R be a commutative ring, let a and b in R, and suppose that aR and bR are comaximal. Show that any g which divides both a and b must be a unit.
(39) Let a and b be relatively prime integers. Show that the ideals $a\mathbb{Z}$ and $b\mathbb{Z}$ are comaximal.
(40) Show that the ideals $xk[x, y]$ and $yk[x, y]$ are **not** comaximal, although the polynomials x and y are relatively prime in $k[x, y]$.

[1] As yet, we have not discussed products of rings, but this problem only considers $R/A \times R/B$ as a set.