Computing Jordan form

This worksheet focuses on how to actually compute the Jordan and generalized Jordan form of a matrix. Let k be a field, V a k-vector space and $A : V \to V$ a linear map. Recall that the ***characteristic polynomial*** of A is the polynomial $\det(t\,\mathrm{Id} - A)$.

(121) Convince yourself that the characteristic polynomial does not depend on a choice of basis for A.

(122) Let k be an algebraically closed field and let A be an $n \times n$ matrix with entries in k. Let the Jordan blocks of A be $J_{n_1}(\lambda_1)$, $J_{n_2}(\lambda_2)$, $\ldots$. Show that the λ_i are the roots of the characteristic polynomial of A.

(123) Continuing the notation of the previous problem, let λ be a root of the characteristic polynomial. Give a formula for $\dim \mathrm{Ker}(\lambda \mathrm{Id} - A)^m$ as a function of m and the n_i.

(124) Consider the following matrix A and its powers

$$A = \begin{bmatrix} -10 & 57 & 6 & -33 & -17 \\ 4 & -21 & -6 & 15 & 5 \\ -10 & 57 & 6 & -33 & -17 \\ 8 & -42 & -12 & 30 & 10 \\ -4 & 21 & 6 & -15 & -5 \end{bmatrix} \quad A^2 = \begin{bmatrix} 72 & -396 & -72 & 252 & 108 \\ 36 & -198 & -36 & 126 & 54 \\ 72 & -396 & -72 & 252 & 108 \\ 72 & -396 & -72 & 252 & 108 \\ -36 & 198 & 36 & -126 & -54 \end{bmatrix} \quad A^3 = 0$$

Compute the Jordan form of A over $\mathbb{C}$.

(125) We no longer assume that k is algebraically closed. Let A be an $n \times n$ matrix with entries in k and let $J_{n_i}(f_i)$ be the generalized Jordan blocks of A. Show that the polynomials $f_i(t)$ are the irreducible factors of the characteristic polynomial of A.

(126) Let f be an irreducible factor of the characteristic polynomial of A. Give a formula for $\dim \mathrm{Ker} f(A)^m$ as a function of m and the n_i.