Euclidean Rings

Vocabulary: Euclidean

Definition. Suppose R is an integral domain. A ***norm*** on R is any function $N\colon R \to \mathbb{Z}_{\geq 0}$. The function N is said to be a ***positive norm*** provided that $N(r) > 0$ for all nonzero r. We call N a ***multiplicative norm*** if $N(ab) = N(a)N(b)$.

Some examples: The normal absolute value on $\mathbb{Z}$ is a positive norm. The norm map $N(a+bi) = a^2+b^2$ on the Gaussian Integers $\mathbb{Z}[i]$ is a positive norm. If k is a field, then we can define a norm on $k[x]$ by $N(p(x)) = \deg p$ for $p \neq 0$ and $N(0) = 0$. We can be a bit more clever and make our norm positive and multiplicative by choosing some positive integer $c \geq 2$ and defining $N(p) = c^{\deg(p)}$ for $p \neq 0$ and $N(0) = 0$.

Definition. An integral domain R is called an ***Euclidean Domain*** provided that there is a positive norm N on R such that for any two elements $a, b \in R$ with $b \neq 0$ there exist q, and $r \in R$ with

$$a = bq + r \text{ and } N(r) < N(b).$$

The element q is called the ***quotient*** and the element r is called the ***remainder*** of the division.

(67) Let k be a field. Show that k is Euclidean with respect to the norm that $N(0) = 0$ and $N(x) = 1$ for $x \neq 0$.

(68) Let k be a field. Verify that $k[x]$ is Euclidean with respect to the norm $N(p) = c^{\deg(p)}$ discussed at the end of the paragraph above.

(69) Let R be an integral domain with positive multiplicative norm N, and let K be its field of fractions. For $\frac{a}{b} \in K$, define $N_K\left(\frac{a}{b}\right) = \frac{N(a)}{N(b)}$.
 (a) Show that $N_K(\ )$ is a well defined function $K \to \mathbb{Q}_{\geq 0}$.
 (b) Show that R is Euclidean with respect to N if and only if, for each $x \in K$, there is an $q \in R$ such that $N_K(x-q) < 1$.

(70) Verify that $\mathbb{Z}[i]$ is Euclidean with respect to the norm $N(a+bi) = a^2+b^2$. Show that q and r need not be unique by considering $a = 3+5i$ and $b = 2$ in $\mathbb{Z}[i]$.

The main use of the Euclidean condition is through the following Theorem:

(71) **IMPORTANT RESULT:** Show that every Euclidean domain is a PID.

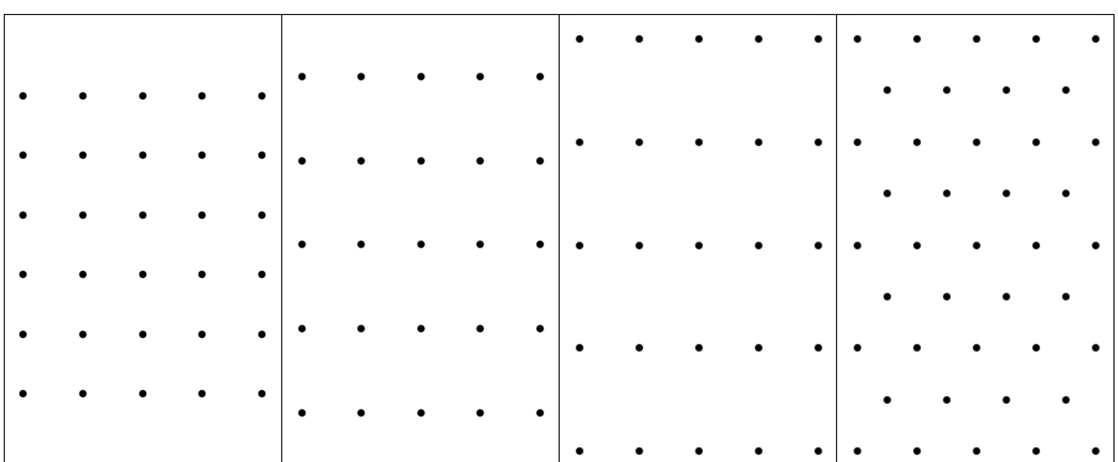
Here are some bonus fun problems about Euclidean domains.

(72) Show that $\mathbb{Z}[\sqrt{-2}]$ is Euclidean, with respect to the norm $N(a+b\sqrt{-2}) = a^2+2b^2$.

(73) Show that $\mathbb{Z}[\sqrt{-3}]$ is **not** Euclidean, with respect to the norm $N(a+b\sqrt{-3}) = a^2+3b^2$, but that $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is Euclidean with respect to the norm $N\left(\frac{c+d\sqrt{-3}}{2}\right) = \frac{c^2+3d^2}{4}$.

(74) Let R be a Euclidean domain. Show that there is some nonunit f such that every nonzero residue class in R/fR is represented by a unit of R. Deduce that $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is not Euclidean for any norm function.

For your convenience, here are scale diagrams of $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-2}]$, $\mathbb{Z}[\sqrt{-3}]$ and $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. In each case, $N(x) = |x|^2$, where $|\ |$ is Euclidean distance measured on the page.



$\mathbb{Z}[i]$	$\mathbb{Z}[\sqrt{-2}]$	$\mathbb{Z}[\sqrt{-3}]$	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$