IDEALS

Vocabulary: Ideal, left ideal, right ideal, two-sided ideal, cyclic module, quotient ring

Definition. Suppose R is a ring. A subset $I \subset R$ is called a ***left ideal*** provided that

I1: $(I, +)$ is a subgroup of $(R, +)$; and
I2: for all $r \in R$ we have $rI \subset I$, that is $rx \in I$ for all $x \in I$.

It is called a ***right ideal*** provided that

I1: $(I, +)$ is a subgroup of $(R, +)$; and
I2: for all $r \in R$ we have $Ir \subset I$, that is $yr \in I$ for all $y \in I$.

A subset of R that is both a left and right ideal is called an ***ideal*** or ***two-sided ideal***.[a]

[a] Professor Speyer's usual practice is to always say "two-sided ideal" in a non-commutative ring. In his experience, this is common among mathematicians who usually work in the commutative world, but who need to deal with non-commutative things occasionally.

If R is commutative, then every left ideal is a right ideal is a two-sided ideal is an ideal.

(20) Show that if A and B are ideals, then $A + B := \{a + b : a \in A, b \in B\}$ is also an ideal.
(21) Let D be a nonsquare integer, let $R = \mathbb{Z}[\sqrt{D}]$ and let p be a prime. Show that R has an ideal I with $|R/I| = p$ if and only if D is a square modulo p.
(22) Fix $n \geq 2$. Let I be the subset of $R = \mathrm{Mat}_{n\times n}(\mathbb{Q})$ consisting of matrices with nonzero entries only in the first row. Is I a left ideal? Is it a right ideal?
(23) Suppose R and S are rings and $\varphi \in \mathrm{Hom}(R, S)$. Show that $\ker(\varphi)$ is a two-sided ideal of R.
(24) Let R be a ring and let I be a left ideal. Since I and R are abelian groups with respect to $+_R$, we can form the quotient group R/I. Show that R/I has a natural structure as a left R-module.

An R-module which can be written in the form R/I is called ***cyclic***.

(25) Let R be a ring and let I be a two sided ideal. Show that R/I has a natural ring structure.