# MODULES

**Vocabulary:** module, module homomorphism, direct sum, endomorphism ring, bilinear
Groups are meant to act on sets. Similarly, rings are meant to act on abelian groups.

**Defintion**. Suppose $R$ is a ring. A ***left $R$-module*** is a set $M$ with two operations:

- $+\colon M \times M \to M$ (called ***addition***) and
- $*\colon R \times M \to M$ (called ***scalar multiplication***)

and an element $0_M$ satisfying the following axioms:

M1: $(M, +, 0_M)$ is an abelian group,
M2: $(r+s)*m = r*m + s*m$ for all $r, s \in R$ and $m \in M$
M3: $(rs)*m = r*(s*m)$ for all $r, s \in R$ and $m \in M$
M4: $r*(m+n) = r*m + r*n$ for all $r \in R$ and $m, n \in M$
M5: $1_R * m = m$ for all $m \in M$.[a]

"$M$ ***is an $R$-module***" will mean "$M$ is a left $R$-module".

[a] As you might guess, some people do not impose this last condition.

The map $*\colon R \times M \to M$ is called an ***action*** of $R$ on $M$ and the elements of $R$ are often called ***scalars***.

(11) Show that $\mathbb{Z}^2$ is a left-$\mathrm{Mat}_{2\times 2}(\mathbb{Z})$-module by having $X \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ act on $\mathbb{Z}^2$ by taking $v \in \mathbb{Z}^2$ to $Xv$.
(12) Formulate the definition of a right $R$-module (the scalar multiplication map will look like $M \times R \to M$). Show that $\mathbb{Z}^2$ is a right-$\mathrm{Mat}_{2\times 2}(\mathbb{Z})$-module by having $X \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ act on $\mathbb{Z}^2$ by taking $v \in \mathbb{Z}^2$ to $X^T v$.
(13) Suppose $R$ is a commutative ring. Show that the polynomial ring $R[x_1, x_2, \ldots, x_n]$ is an $R$-module. Do we need to assume that $R$ is commutative?
(14) What happens if we don't assume that $+_M$ is commutative? Hint: Consider $(1_R + 1_R) * (u +_M v)$.

**Definition**. Suppose $R$ is a ring and $M$ and $N$ are $R$-modules. A function $g\colon M \to N$ is called an ***$R$-module homomorphism*** provided that

- $g$ is a group homomorphism and
- $g(rm) = rg(m)$ for all $r \in R$ and $m \in M$.

The set of $R$-module homomorphisms from $M$ to $N$ is denoted $\mathrm{Hom}_R(M, N)$ or $\mathrm{Hom}_{R-\mathrm{mod}}(M, N)$. We set $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ and call $\mathrm{End}_R(M)$ the ***endomorphism ring of*** $M$.

(15) Suppose $R$ is a commutative ring and $M$ is an $R$-module. Show that there is a "natural" map of rings $R \to \mathrm{End}_R(M)$. What if $R$ is not commutative?

**Definition**. Suppose $R$ is a ring and $M$ and $N$ are $R$-modules. The ***direct sum*** of $M$ and $N$, written $M \oplus N$, is the $R$-module defined as follows: An element of $M \oplus N$ is an ordered pair $(m, n)$ with $m \in M$ and $n \in N$. We have $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$, $r(m, n) = (rm, rn)$.

(16) Check that $M \oplus N$ is an $R$-module.
(17) Let $M_1$, $M_2$, $M$, $N_1$, $N_2$ and $N$ be $R$-modules. Show that $\mathrm{Hom}_R(M_1 \oplus M_2, N) \cong \mathrm{Hom}_R(M_1, N) \times \mathrm{Hom}_R(M_2, N)$ and $\mathrm{Hom}_R(M, N_1 \oplus N_2) \cong \mathrm{Hom}_R(M, N_1) \times \mathrm{Hom}_R(M, N_2)$ as abelian groups.
(18) Let $L_1, L_2, \ldots, L_p$, $M_1, M_2, \ldots, M_q$ and $N_1, N_2, \ldots, N_r$ be $R$-modules, and let $L = \bigoplus L_i$, $M = \bigoplus M_j$ and $N = \bigoplus N_k$. Describe a way to write elements of $\mathrm{Hom}_R(L, M)$, $\mathrm{Hom}_R(M, N)$ and $\mathrm{Hom}_R(L, N)$ as matrices, so that the composition map $\mathrm{Hom}_R(L, M) \times \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(L, N)$ corresponds to matrix multiplication.

**Defintion**. Let $R$ be a commutative ring and let $A$, $B$ and $C$ be $R$-modules. A map $\langle\ ,\ \rangle : A \times B \to C$ is called ***bilinear*** or $R$-***bilinear*** if

- $\langle a_1 + a_2, b\rangle = \langle a_1, b\rangle + \langle a_2, b\rangle$.
- $\langle a, b_1 + b_2\rangle = \langle a, b_1\rangle + \langle a, b_2\rangle$.
- $r * \langle a, b\rangle = \langle r * a, b\rangle = \langle a, r * b\rangle$.

where $a$, $a_1$ and $a_2$ are in $A$, $b$, $b_1$ and $b_2$ are in $B$, and $r$ is in $0$.

(19) In the above definition, label every $+$ and $*$ with a subscript $A$, $B$ or $C$ as appropriate. This sort of abuse of notation is frequent in algebraic writing; you should get used to it, but you should also always be able to fill in the missing symbols if needed.