

## NOETHERIAN RINGS

**Vocabulary:** Noetherian, left Noetherian, right Noetherian

Noetherianity is basically a condition to prevent things from getting “too infinite”. For now, we’ll prove the equivalence of various definitions of Noetherianity but leave without many examples. Also, we’ll work with non-commutative rings on this worksheet, but we’ll wind up mostly using the concept in the commutative case.

Let  $R$  be a ring. Consider the following conditions on  $R$ .

Condition 1(a): Every left ideal  $I$  of the ring  $R$  is finitely generated.

Condition 2(a): For any chain of left ideals  $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$  of  $R$ , we have  $I_r = I_{r+1}$  for all sufficiently large  $R$ .

Condition 3(a): Given any nonempty collection  $\mathcal{X}$  of left ideals of  $R$ , there is some  $I \in \mathcal{X}$  which is not contained in any other  $I' \in \mathcal{X}$ .

Condition 1(b): Every left  $R$ -submodule  $M$  of  $R^n$  is finitely generated.

Condition 2(b): For any chain of left  $R$ -submodules  $M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$  of  $R^n$ , we have  $M_r = M_{r+1}$  for all sufficiently large  $R$ .

Condition 3(b): Given any nonempty collection  $\mathcal{X}$  of left  $R$ -submodules of  $R^n$ , there is some  $M \in \mathcal{X}$  which is not contained in any other  $M' \in \mathcal{X}$ .

Condition 1(c): For any finitely generated left  $R$ -module  $S$ , every left  $R$ -submodule  $M$  of  $S$  is finitely generated.

Condition 2(c): For any finitely generated left  $R$ -module  $S$ , for any chain of left  $R$ -submodules  $M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$  of  $S$ , we have  $M_r = M_{r+1}$  for all sufficiently large  $R$ .

Condition 3(c): For any finitely generated left  $R$ -module  $S$ , given any nonempty collection  $\mathcal{X}$  of left  $R$ -submodules of  $S$ , there is some  $M \in \mathcal{X}$  which is not contained in any other  $M' \in \mathcal{X}$ .

(49) Prove all these definitions are equivalent.<sup>1</sup>

**Definition.** A ring which obeys these conditions is called *left Noetherian*. A ring which obeys these conditions with “right” in place of “left” is called *right Noetherian*. A ring which is left and right Noetherian is called Noetherian<sup>a</sup>.

<sup>a</sup>Named for Emmy Noether, German mathematician 1882-1935, who has a decent case for being the greatest algebraist of all time.

<sup>1</sup>If you don’t assume the Axiom of Choice, then the conditions in each column are still equivalent to each other, and the implications  $3(x) \implies 1(x) \implies 2(x)$  still hold, but I don’t know about the reverse implications. However, the use of Choice in showing  $2(x) \implies 3(x)$  is very simple.