Classification of finitely generated modules over a PID

Throughout this worksheet, let R be a PID.

(103) Let $X \in \mathrm{Mat}_{m\times n}(R)$ and let $(d_1, d_2, \ldots, d_{\min(m,n)})$ be the invariant factors of X.

   (a) Show that $R^m/XR^n \cong \bigoplus R/d_jR \oplus R^{m-\min(m,n)}$.

   (b) Show that $\mathrm{Ker}(X) \cong R^{\#\{j : d_j = 0\}+n-\min(m,n)}$.

(104) Let S be a ring and let M be a finitely generated S-module.

   (a) Show that there is a surjection $S^{\oplus m} \twoheadrightarrow M$ for some m.

   (b) Suppose that S is Noetherian (for example, every PID is Noetherian). Show that there is a surjection $S^n \twoheadrightarrow \mathrm{Ker}(S^m \to M)$ for some n.

   (c) With hypotheses and assumptions as in the previous part, show that there is an $m \times n$ matrix X with $M \cong S^m/XS^n$.

(105) **(Classification of modules over a PID: Elementary divisor form)** Show that every finitely generated R-module M is of the form $\bigoplus R/d_jR$ for some nonunits $d_1, d_2, \ldots, d_k$ in R with $d_1|d_2|\cdots|d_k$.

(106) **(Classification of modules over a PID: Prime power form)** Show that every finitely generated R-module M is of the form $R^{\oplus r} \oplus \bigoplus R/p_j^{e_j}R$ for some nonnegative integer r, some sequence of prime elements p_j and some sequence of positive integers e_j.

(107) Let M be a finitely generated R-module.

   (a) Show that the $d_1, d_2, \ldots, d_k$ in Problem 105 are unique up to multiplication by units.

   (b) Show that the r, p_j and e_j in Problem 106 are unique up to rearrangement and up to multiplying the p_j by units.

   Hint for both parts: One approach is to study M/qM for various choices of q.

Let's see what this say for some particular PID's:

(108) Let k be a field, then k is also a PID. What have we proved about finitely generated k-modules?

(109) A $\mathbb{Z}$-module is the same thing as an abelian group.

   (a) What have we proved about finitely generated abelian groups?

   (b) Consider the matrices below as maps $\mathbb{Z}^n \to \mathbb{Z}^m$. Describe their cokernels:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$