PRINCIPAL IDEAL DOMAINS (PIDS)

Vocabulary: Principal ideal, principal ideal domain (PID)

Definition. Suppose R is a ring and $I \subset R$ is a two-sided ideal. I is called ***principal*** provided that $I = (r)$ for some $r \in R$.

(75) Show that every ideal in $\mathbb{Z}$ is principal. Do **not** assume unique factorization into primes. (Hint: Take the smallest positive element of the ideal.)

(76) Let R be an integral domain and let X be a subset of R and let I be the ideal generated by X. Show that, if $I = (g)$ for some $g \in R$, then g is a GCD of X.

Definition. A ***Principal Ideal Domain*** or ***PID*** is an integral domain in which every ideal is principal.

So Problem 76 shows that, in a PID, every set has a GCD.

(77) Show that every PID is Noetherian.

(78) Show that a PID is a UFD.[1] (Hint: This is almost entirely citations to the UFD worksheet.)

We note in particular that we have now shown $\mathbb{Z}$ is a UFD.

(79) Suppose R is a PID. Show that every nonzero prime ideal in R is a maximal ideal.

We conclude with some fun and useful lemmas about matrices over PID's:

(80) Let R be any commutative ring and let x and y be two elements of R with (x) and (y) comaximal.
  (a) Show that there is a 2×2 matrix $\left[\begin{smallmatrix} x & u \\ y & v \end{smallmatrix}\right]$ with entries in R and determinant 1.
  (b) Show that there is a 2×2 matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with entries in R and determinant 1 such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(81) Let R be a PID and let x and $y \in R$. Show that there is a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with entries in R and determinant 1 and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathrm{GCD}(x,y) \\ 0 \end{bmatrix}.$$

(82) Let R be a PID and let x and $y \in R$. Show that there are 2×2 matrices U and V with entries in R and determinant 1 such that:

$$U \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} V = \begin{bmatrix} \mathrm{GCD}(x,y) & 0 \\ 0 & \mathrm{LCM}(x,y) \end{bmatrix}.$$

Here $\mathrm{LCM}(x,y) := \frac{xy}{\mathrm{GCD}(x,y)}$.

[1] This need not hold without Choice; Hodges, "Luchli's algebraic closure of Q", *Proceedings of the Cambridge Philosophical Society*, 1976 showed that it is consistent with ZF for there to be a PID in which some elements have no factorization into irreducibles.