Problem Set 1: (Due Friday September 13)

Please see the course website for policy regarding collaboration and formatting your homework.

(1) Suppose R is a ring and $a, b \in R$. Show that, if b is a unit and a divides b on both the left and the right, then a is a unit.

(2) Let K be a field and let R be a subring of K. Let S be a nonempty subset of R, closed under multiplication and not containing 0. Let $S^{-1}R$ be the set of elements in K which can be written as $\frac{a}{b}$ with $a \in R$ and $b \in S$. Show that $S^{-1}R$ is a subring of K.

(3) Let A be a ring. The ***center*** of A is $Z(A) := \{z \in A : az = za \text{ for all } a \in A\}$. Show that $Z(A)$ is a subring of A.

(4) What is the cardinality of the following rings?
  (a) $\mathbb{Z}[x]/(6, 2x-1)$.
  (b) $\mathbb{Z}[x]/(x^2-3, 2x+4)$.

(5) Let k be a field. Show that the only ideals of k are (0) and k.

(6) Let k be a field.
  (a) Describe all left ideals of in the ring $\mathrm{Mat}_{n\times n}(k)$ of $n \times n$ matrices with entries in k. Hint: Row reduction.
  (b) Show that the only two-sided ideals of $\mathrm{Mat}_{n\times n}(k)$ are (0) and $\mathrm{Mat}_{n\times n}(k)$.

(7) Let R be the set of all infinite sequences $(x_1, x_2, \dots,)$ in $\mathbb{R}$ for which $\lim_{n\to\infty} x_n$ exists. We define addition and multiplication on R by $(x_j) + (y_j) = (x_j + y_j)$ and $(x_j)(y_j) = (x_j y_j)$.
  (a) Show that R is a commutative ring.
  (b) Let $\mathfrak{m}$ be the set of sequences (x_j) for which $\lim_{n\to\infty} x_n = 0$. Show that $\mathfrak{m}$ is a maximal ideal of R.

(8) Suppose k is a field and let $R = k[t]$. Show that R-modules are "the same" as k-vector spaces V equipped with a k-linear endomorphism $T : V \to V$. This question can be interpreted in two ways:
  - (for those who don't know what categories are) Give a bijection between isomorphism classes of R-modules and isomorphism classes of pairs (V, T); this includes defining when (V_1, T_1) and (V_2, T_2) are isomorphic.
  - (for those who know what categories are) Define the category of R-modules and the category of pairs (V, T), and give an equivalence between them.

(9) Suppose k is a commutative ring. An ***k-algebra*** is an k-module A with an k-bilinear map $A \times A \to A$.

Let k be a commutative ring and let A be any ring. Give a bijection between ways to consider A as an k-algebra, and ring maps $k \to Z(A)$. (This problem includes defining what "ways to consider A as an k-algebra" means.)

(10) Suppose R is a ring. An element $e \in R$ is called ***idempotent*** if $e^2 = e$.
  (a) Give an example of an idempotent, other than 0 and 1, in $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$.
  (b) Give an example of an idempotent, other than 0 and 1, in $\mathbb{Z}/15\mathbb{Z}$.
  (c) Let e be an idempotent of R and let $eRe = \{ere : r \in R\}$. Show that eRe is a ring, with respect to the addition and multiplication operations of R, where $0_{eRe} = 0_R$ and $1_{ere} = e$.

(11) Let R be a ring. For $r \in R$ and $1 \le i \ne j \le n$ define the $n \times n$ matrix $E(i, j, r)$ by

$$E(i,j,r)_{k\ell} = \begin{cases} 1 & \text{if } k = \ell, \\ r & \text{if } k = i \text{ and } j = \ell, \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $E(i, j, r)$ is known as an ***elementary matrix***.
  (a) Suppose X is an $m \times n$ matrix. What is the effect of right multiplication by $E(i, j, r)$ on X? Suppose Y is an $n \times m$ matrix. What is the effect of left multiplication by $E(i, j, r)$ on Y? What is the inverse of $E(i, j, r)$?
  (b) Show that the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ is a product of 2×2 elementary matrices.
  (c) Let u be a unit of R. Show that the matrix $\left[\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right]$ is a product of 2×2 elementary matrices.

Additional task: We will use Zorn's Lemma in this class. Please familiarize yourself with its statement and with how it is used to show that every nonzero ring has a maximal ideal. Good references are pages 907-909 in Dummit and Foote, Keith Conrad's notes `https://kconrad.math.uconn.edu/blurbs/zorn1.pdf` and Dan Grayson's proof at `https://faculty.math.illinois.edu/~dan/ShortProofs/Zorn.pdf`.