Problem Set 10 (Due **Monday**, December 9)

(75) Compute the following matrices in the obvious bases for the vector spaces involved:

$$(a)\ \left[\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right]\otimes\left[\begin{smallmatrix}5&6\\7&8\end{smallmatrix}\right] \qquad (b)\ \mathrm{Sym}^4\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right] \qquad (c)\ \mathrm{Alt}^2\left[\begin{smallmatrix}1&1&1\\1&2&3\\1&3&6\end{smallmatrix}\right]$$

(76) Recall that the rank of a linear map $\phi : V \to W$ is the dimension of $\phi(V)$. Show that $\mathrm{Alt}^k \phi = 0$ if and only if the rank of ϕ is $< k$. Please don't use that the rank of a matrix is the size of its largest nonvanishing minor.

(77) Let k be a field and let V and W be k-vector spaces. Let $V^\vee$ be the dual vector space to V.
  - (a) Show that there is a linear map $\phi : V^\vee \otimes_k W \to \mathrm{Hom}_k(V, W)$ such that $\phi(\lambda \otimes w)(v) = \lambda(v)w$.
  - (b) Show that the image of ϕ is precisely the linear maps $V \to W$ of finite rank. In particular, if $\dim V$ or $\dim W < \infty$, show that ϕ is surjective.
  - (c) Show that every element of $V^\vee \otimes_k W$ can be represented in the form $\sum_{j=1}^n \lambda_j \otimes w_j$ where $w_1, w_2, \ldots, w_n$ are linearly independent.
  - (d) Show that ϕ is always injective. Hint: Write an element of the kernel in the form from Problem (77c).

(78) In this problem, we will classify alternating bilinear forms on a finite dimensional vector space up to change of basis. Let k be a field, V a k-vector space and $\langle\ ,\ \rangle$ an alternating form on V.
  - (a) Show that, if $\langle\ ,\ \rangle \neq 0$, there is a 2-dimensional subspace L of V such that $\langle\ ,\ \rangle$ restricts to a nondegenerate bilinear form on L.
  - (b) Let X be an $n \times n$ matrix with $X_{ij} = -X_{ji}$ and $X_{ii} = 0$. Show that there is an invertible matrix S such that SXS^T is of the form

$$\begin{bmatrix} \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & & & \\ & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

    for some number of $\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$ blocks and some number of 0's.
  - (c) Find such an S for $k = \mathbb{Q}$ and $X = \left[\begin{smallmatrix}0&1&1&1\\-1&0&1&1\\-1&-1&0&1\\-1&-1&-1&0\end{smallmatrix}\right]$.
  - (d) Show that every alternating matrix has even rank.
  - (e) Show that the determinant of an alternating matrix is always a square in k. The square root of $\det X$ (defined up to sign) is called the ***Pfaffian*** of X.

(79) We defined $\mathrm{Sym}^d V$ as a quotient of $V^{\otimes d}$. Let $\mathrm{Sym}_d V \subseteq V^{\otimes d}$ be those tensors invariant under permutation of tensor factors. Some books define this subspace to be $\mathrm{Sym}^d V$ instead of the quotient that we use.
  - (a) Let V have basis $e_1, e_2, \ldots, e_n$. Give a basis of $\mathrm{Sym}_d V$ and show that $\dim \mathrm{Sym}_d V = \dim \mathrm{Sym}^d V$.
  - (b) For any linear map $\phi : V \to W$, define a linear map $\mathrm{Sym}_d \phi : \mathrm{Sym}_d V \to \mathrm{Sym}_d W$ such that the diagram

$$\begin{array}{ccccc} \mathrm{Sym}_d V & \longrightarrow & V^{\otimes d} & \longrightarrow & \mathrm{Sym}^d V \\ \downarrow{\scriptstyle \mathrm{Sym}_d \phi} & & \downarrow{\scriptstyle \phi^{\otimes d}} & & \downarrow{\scriptstyle \mathrm{Sym}^d \phi} \\ \mathrm{Sym}_d W & \longrightarrow & W^{\otimes d} & \longrightarrow & \mathrm{Sym}^d W \end{array} \qquad \text{commutes.}$$

  - (c) If the characteristic of k is either 0 or else a prime $p > d$, show that the composition $\mathrm{Sym}_d V \hookrightarrow V^{\otimes d} \twoheadrightarrow \mathrm{Sym}^d V$ is an isomorphism.

  Now, let k be a field with characteristic p.
  - (d) Show that $\mathrm{Sym}^p\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = \mathrm{Sym}^p\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \mathrm{Sym}^p\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ but $\mathrm{Sym}_p\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \neq \mathrm{Sym}_p\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \mathrm{Sym}_p\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.
  - (e) Show that there is no choice of isomorphisms such that

$$\begin{array}{ccc} \mathrm{Sym}_p V & \xrightarrow{\cong} & \mathrm{Sym}^p V \\ \downarrow{\scriptstyle \mathrm{Sym}_p \phi} & & \downarrow{\scriptstyle \mathrm{Sym}^p \phi} \\ \mathrm{Sym}_p W & \xrightarrow{\cong} & \mathrm{Sym}^p W \end{array}$$

    commute for all $\phi : V \to W$. You have shown that the functors Sym_p and Sym^p are not isomorphic.

(80) Enjoy your winter break! This problem is due on January 8.