Problem Set 3 (Due Friday September 27)

Please see the course website for policy regarding collaboration and formatting your homework.

(20) Let R be a commutative ring and P a prime ideal such that R/P is finite. Show that P is maximal.

(21) Let R be a finite ring, and let $|R|$ factor as $\prod p^{a_p}$. Show that there are rings R_p, for the various primes p dividing N, such that $|R_p| = p^{a_p}$ and $R \cong \prod R_p$.

(22) As promised, we revisit Problem (14) without the hypothesis that the elements of S are not zero divisors. Let R be a commutative ring and let S be a subset of R which is closed under multiplication and contains 1. Define a relation $\sim$ on $S \times R$ by $(s, r) \sim (s', r')$ if there is an element s'' of S such that $s''sr' = s''s'r$.

(a) Show that $\sim$ is an equivalence relation.

(b) Show that the simpler definition, $(s, r) \approx (s', r')$ if $sr' = s'r$, need not be an equivalence relation.

As before, we define $S^{-1}R$ to be $S \times R/\sim$. I won't make you write it out but, once again, $S^{-1}R$ is a ring.

(c) Give a simple description of $\{3^k : k \in \mathbb{Z}_{\geq 0}\}^{-1}(\mathbb{Z}/15\mathbb{Z})$.

(d) Show that the kernel of the map $r \mapsto 1^{-1}r$ from R to $S^{-1}R$ is $\{x \in R : \exists_{s \in S}\ sx = 0\}$.

(e) State and prove a universal property of $S^{-1}R$. In other words, your statement should look like "Given a commutative ring R', and the following additional data ..., there is a unique map $S^{-1}R \to R'$ such that ...."

(23) Let R be a commutative ring and let P be a prime ideal of R.

(a) Show that $R \setminus P := \{x \in R : x \notin P\}$ is closed under multiplication.

We define $R_P := (R \setminus P)^{-1}R$ and call R_P the ***local ring*** of P.

(b) Justify this terminology by showing that R_P is a local ring (see problem 15).

(24) An element x in a ring R is called ***nilpotent*** if there is a positive integer N such that $x^N = 0$.

(a) Show that, if x is nilpotent, then $1 + x$ is a unit.

(b) Show that, if x and y are nilpotents with $xy = yx$, then $x + y$ is nilpotent. Is this true if x and y do not commute?

(c) Show that if R is commutative, then the set of nilpotent elements in R form an ideal; it is called the ***nilradical of*** R and often denoted $\mathrm{Nil}(R)$.

(25) Let R be a UFD.

(a) Let A and B be matrices with entries in R, of sizes $r \times s$ and $s \times t$ respectively, and let $C = AB$; we write A_{ij}, B_{jk} and C_{ik} for the entries of these matrices. Prove or disprove: $\mathrm{GCD}(C_{ik}) = \mathrm{GCD}(A_{ij})\,\mathrm{GCD}(B_{jk})$. (The left hand side is the GCD of all entries of C, and similarly for A and B.)

(b) Let $a(x) = \sum a_i x^i$ and $b(x) = \sum b_j x^j$ be polynomials with coefficients in R and let $c(x) = a(b)b(x) =: \sum c_k x^k$. Prove or disprove: $\mathrm{GCD}(c_k) = \mathrm{GCD}(a_i)\,\mathrm{GCD}(b_j)$. (The left hand side is the GCD of all coefficients of $c(x)$, and similarly for $a(x)$ and $b(x)$.)

(26) (a) Show that the ring of integers, $\mathbb{Z}$, is Noetherian.

(b) Any ring which contains a subfield k and is finite dimensional as a k-vector space. Here take the scalar multiplication from k by left multiplication, and show that R is left-Noetherian.

(c) Show that the polynomial ring $k[x]$ is Noetherian, for any field k.

(27) Show that the following rings are **not** Noetherian:

(a) The polynomial ring $k[x_1, x_2, \ldots]$ in infinitely many variables.

(b) $k[x, x^{1/2}, x^{1/4}, x^{1/8}, \cdots]$, for k any field. An element of this ring is a formal finite sum $\sum a_j x^{b_j/2^{n_j}}$.

(c) The subring of $k[x, y]$ generated by all monomials of the form $x^j y$.

(28) Let R be a Noetherian ring and let I be a two-sided ideal. Show that R/I is Noetherian.

(29) This problem provides a proof of Hilbert's basis theorem, which states: If R is a Noetherian commutative ring, then $R[x]$ is Noetherian. Problems 26c, 26a and 28, together with Hilbert's basis theorem, show that any commutative ring which is finitely generated over k or $\mathbb{Z}$ is Noetherian.

Let I be an ideal of $R[t]$; we will show that I is finitely generated. Define I_d to be the set of $f \in R$ such that there is an element of I of the form $fx^d + f_{d-1}x^{d-1} + \cdots + f_1 t + f_0$.

(a) Show that I_d is an ideal of R and that $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$.

(b) Show that there is an index r such that $I_r = I_{r+1} = I_{r+2} = \cdots$. Show that I_r is finitely generated.

(c) Let M be the set of polynomials in I with degree $\leq r$. Show that M is finitely generated as an R-module.

Let $f_1, f_2, \ldots, f_m$ generate I_r as an R-module and choose elements g_j of I of the form $g_j = f_j x^r +$(lower order terms). Let $h_1, h_2, \ldots, h_n$ generate M as an R-module.

(d) Show that $g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n$ generate I as an $R[x]$ module.