Problem Set 6 (Due Friday, October 25)

(47) Set

$$M = \begin{bmatrix} 2 & 4 & 10 \\ 1 & 3 & 7 \\ 1 & 1 & 15 \end{bmatrix}.$$

Let G be the abelian group $\mathbb{Z}^3/M\mathbb{Z}^3$. Write G as a product of cyclic groups of prime power order.

(48) Suppose that a and b are integers and consider the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ given by $(x, y) \mapsto (2x + 2y, ax + by)$. Let H be the image of this map and let $G = \mathbb{Z}^2/H$.
 (a) For which a and b is G isomorphic to $\mathbb{Z}$?
 (b) For which a and b is G isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$?

(49) Let k be a field. Show that there do **not** exist 2×2 invertible matrices in the ring $k[x, y]$ with

$$U \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} V = \begin{bmatrix} xy & 0 \\ 0 & 1 \end{bmatrix}.$$

(50) Let R be a commutative ring and let A be an $n \times n$ matrix with entries in R. In this problem, we will show that the following are equivalent: A has a left inverse, A has a right inverse, and $\det A$ is a unit of R.
 (a) Show that, if there is an $n \times n$ matrix B with entries in R such that $AB = \mathrm{Id}_n$, then $\det A$ is a unit. Show that the same conclusion holds if we assume $BA = \mathrm{Id}_n$ instead.
 (b) Let $\widehat{A}_{ij}$ be the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A. The ***adjugate matrix*** of A is the $n \times n$ matrix $\mathrm{adj}(A)$ where the entry in row i and column j is $(-1)^{i+j} \det \widehat{A}_{ji}$. (Note that the i and j are switched!) Prove that

$$A \,\mathrm{adj}(A) = \mathrm{adj}(A)\, A = (\det A)\mathrm{Id}_n.$$

 Hint: If you've never heard of cofactor expansion, look it up now.
 (c) Show that, if $\det A$ is a unit of R, then the matrix A has an inverse given by an $n \times n$ matrix with entries in R.

(51) Let R be a commutative integral domain. We remind the reader that a ***short exact sequence*** of R-modules is a sequence $0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0$ where M_1, M_2, M_3 are R-modules, α and β are maps of R-modules, α is injective, β is surjective and $\mathrm{Image}(\alpha) = \mathrm{Ker}(\beta)$.
 (a) Let A, B and C be matrices with entries in R, of sizes $m_1 \times n_1$, $m_1 \times n_2$ and $m_2 \times n_2$ respectively. Assume that $A : R^{n_1} \to R^{m_1}$ and $C : R^{n_2} \to R^{m_2}$ are injective. Show that there is a short exact sequence:

$$0 \to R^{m_1}/AR^{n_1} \to R^{m_1+m_2}/\left[\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right] R^{n_1+n_2} \to R^{m_2}/CR^{n_2} \to 0.$$

 (b) Let A and C be matrices with entries in R of sizes $k_2 \times k_1$ and $k_3 \times k_2$. Show that there are invertible matrices U and V of sizes $(k_2 + k_3) \times (k_2 + k_3)$ and $(k_1 + k_2) \times (k_1 + k_2)$ such that

$$\begin{bmatrix} CA & 0 \\ 0 & \mathrm{Id}_{k_2} \end{bmatrix} = U \begin{bmatrix} A & \mathrm{Id}_{k_2} \\ 0 & C \end{bmatrix} V.$$

 (c) Let A and C be matrices with entries in R of sizes $k_2 \times k_1$ and $k_3 \times k_2$. Assume that $A : R^{k_1} \to R^{k_2}$ and $C : R^{k_2} \to R^{k_3}$ are injective. Show that there is a short exact sequence:

$$0 \to R^{k_2}/AR^{k_1} \to R^{k_3}/(CA)R^{k_1} \to R^{k_3}/CR^{k_2} \to 0.$$

This problem set is intentionally short in order to give you time to study for the exam on October 30.

The exam will consist of:

- Proofs of important results from the worksheets up through the proof of the Smith Normal Form Theorem. Expect to see some subset of the following worksheet problems, perhaps slightly rephrased:

  26, 28, 29, 30, 33, 34, 35, 36, 37, 43, 44, 45, 46, 50, 54, 56, 58, 71, 76, 77, 78, 79, 99, 101, 102

- Basic computations involving the Chinese Remainder Theorem, Euclidean Algorithm and Smith Normal Form. Good problems to review are homework problems 13(c), 17(a), 36, 39, 40, 44 and 47 and worksheet problems 97, 109.

Material on Jordan Normal Form will not appear on this exam, as I worry that we may not get to it in time.