Problem Set 7 (Due Friday, November 8)

Remark: This problem set has a number of problems on older material which didn't fit until now. Don't go looking to connect everything to the newest stuff.

(52) Let k be a field and X an $n \times n$ matrix over k. Let I be $\{g(x) \in k[x] : g(X) = 0\}$. Show that $I = (g)$ for some $g(x) \in k[x]$. This g is called the ***minimal polynomial*** of X. We generally normalize g to be monic.

(53) Let k be a field, let $\lambda_1, \lambda_2, \dots, \lambda_r$ be elements of k and let B be a positive integer. Show that there is a polynomial $g(x) \in k[x]$ such that $g(x) \equiv \lambda_j \bmod (x - \lambda_j)^B$ for $1 \leq j \leq r$.

(54) Suppose that A is a 5×5 complex matrix with minimal polynomial $A^5 - A^3$.
  (a) What is the Jordan form of A?
  (b) What is the characteristic polynomial of A^2?
  (c) What is the minimal polynomial of A^2?

(55) Let k be an algebraically closed field. Let X be an $n \times n$ matrix with entries in k, and let the Jordan blocks of X be $J_{n_1}(\lambda_1)$, $J_{n_2}(\lambda_2)$, $\dots$, $J_{n_r}(\lambda_r)$. Express the following quantities in terms of the λ_j and n_j. (Some of you did some of this in your groups, but please repeat it here if you did.):
  (a) The characteristic polynomial of X.
  (b) The minimal polynomial of X, meaning the lowest degree polynomial $g(x)$ such that $g(X) = 0$.
  (c) The dimension of $\mathrm{Ker}(X - \lambda \mathrm{Id})$.

(56) Let R be a UFD and let $S \subset R$ be a set containing 1, not containing 0 and closed under multiplication. In this problem, we will show that $S^{-1}R$ is a UFD. You may want to use the description of $S^{-1}R$ from Homework Problem (2).
  (a) Let P be the set of primes dividing some element of S and let T be the set of products of primes in P (including the empty product, 1). Show that $S^{-1}R \cong T^{-1}R$.
  (b) Let p be prime in R. Show that p is either prime or a unit in $T^{-1}R$.
  (c) Let q be a prime of $T^{-1}R$. Show that q is of the form up where p is a prime of R and u is a unit of $T^{-1}R$.
  (d) Show that $T^{-1}R$ is a UFD.

(57) Let R be a PID. Let M be an $m \times n$ matrix with entries in R. Let X be the set of all elements of R which are of the form $\vec{x}^T M \vec{y}$ where $\vec{x} \in R^m$ and $\vec{y} \in R^n$. Show that X is an ideal of R.

(58) Let R be an integral domain such that, for every nonzero ideal I, the quotient R/I is finite. Let A be an $n \times n$ matrix with entries in R. In this problem, we will show that $|R^n/AR^n| = |R/(\det A)R|$. Thanks to "Max" at `https://math.stackexchange.com/questions/3389832` for suggesting this approach. We write K for $\mathrm{Frac}(R)$. For $A \in \mathrm{Mat}_{n\times n}(R)$ a matrix with $\det A \neq 0$, define $D_n(A) = |R^n/AR^n|$.
  (a) Show that $D_n(AC) = D_n(A)D_n(C)$. Hint: Look at homework problem 51c.
  (b) For $A \in \mathrm{GL}_n(K)$, let M be a nonzero element of R such MA has entries in R. Show that the rational number $M^{-n}D_n(MA)$ does not depend on the choice of M. Define $D_n(A) = M^{-n}D_n(MA)$, and show that we have $D_n(AC) = D_n(A)D_n(C)$ for matrices A and C in $\mathrm{GL}_n(K)$.
  (c) For E an elementary matrix with entries in K, show that $D_n(E) = 1$. (In other words, $E = E(i, j, r)$ for $r \in K$, as in problem 11.) Remember, we don't know that the off diagonal entry of E is in R.
  (d) For a diagonal matrix T with entries in K, show that $D_n(T) = D_1([\det T])$. To be clear, $[\det T]$ is the 1×1 matrix whose entry is $\det T$.
  (e) Show that $D_n(A) = D_1([\det A])$ for any $A \in \mathrm{GL}_n(K)$.

(59) Let R be a ring. A left R-module S is called ***simple*** if $S \neq 0$ and S has no submodules other than 0 and S. A left R-module M is said to have ***finite length*** if there is a chain of submodules $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\ell = M$ such that M_{i+1}/M_i is simple for $0 \leq i < \ell$.
  (a) Suppose that R is an associative unital k-algebra for some field k[1] and let M is finite dimensional as a k-vector space. Show that M is a finite length R-module. (Hint: Choose a chain $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\ell = M$ with ℓ as large as possible, and remember to justify that there is a maximum such ℓ.)

  In general, "finite length" can be thought of as a generalization of "finite dimensional vector space". Let $0 \to K \to M \to Q \to 0$ be a short exact sequence of R-modules.
  (b) Suppose that K and Q have finite length. Show that M has finite length.
  (c) Suppose that M has finite length. Show that K has finite length.
  (d) Suppose that M has finite length. Show that Q has finite length.

[1] In other words, (see problem 9) let R be a ring, let k be a field which is also a subring of the center of R.