Problem Set 8 (Due Friday, November 15)

Please see the course website for policy regarding collaboration and formatting your homework.

(60) Consider the nilpotent matrix whose powers are shown below:

$$X = \begin{bmatrix} -1 & -7 & -2 & -2 \\ 0 & 3 & 1 & 1 \\ 0 & -6 & -2 & -2 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad X^2 = \begin{bmatrix} -1 & -4 & -1 & -1 \\ 1 & 4 & 1 & 1 \\ -2 & -8 & -2 & -2 \\ -1 & -4 & -1 & -1 \end{bmatrix} \quad X^3 = 0$$

$$\mathrm{rank}(X) = 2 \qquad \mathrm{rank}(X^2) = 1 \qquad \mathrm{rank}(X^3) = 0$$

(a) Compute the Jordan normal form of X. Suppose that S is a matrix with columns $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ such that $S^{-1}XS$ is in Jordan normal form, with the larger blocks sorted before the smaller ones. Express the following quantities in terms of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$:

(b) $\mathrm{Ker}(X)$ and $\mathrm{Ker}(X^2)$.

(c) $\mathrm{Im}(X)$ and $\mathrm{Im}(X^2)$.

(d) The intersections $\mathrm{Im}(X) \cap \mathrm{Ker}(X)$, $\mathrm{Im}(X) \cap \mathrm{Ker}(X^2)$, $\mathrm{Im}(X^2) \cap \mathrm{Ker}(X)$ and $\mathrm{Im}(X^2) \cap \mathrm{Ker}(X^2)$.

(61) Let A be an $n \times n$ complex matrix. Show that A is nilpotent if and only if A and $2A$ are similar.

(62) Suppose that A is a 7×7 complex matrix that obeys the polynomial relation $A^5 = 2A^4 + A^3$. Given that the rank of A is 5 and the trace is 4, what is the Jordan canonical form of A?

(63) We didn't get to this one in class, so please do it now. Let $K \subset L$ be fields and let X be an $n \times n$ matrix with entries in K. Show that X has the same rational canonical form over K and over L.

(64) Let k be a field. A square matrix X with entries in k is called ***diagonalizable*** if it is similar to a diagonal matrix.

(a) Show that, if a matrix X is both diagonalizable and nilpotent, then it is zero.

(b) Let X and Y be square matrices with entries in k and let Z be the block matrix $\left[\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right]$. Show that Z is diagonalizable if and only if X and Y are both diagonalizable.

(c) Let X and Y be diagonalizable square matrices which commute with each other. Show that there is an invertible matrix S such that SXS^{-1} and SYS^{-1} are both diagonal. Hint: Start by reducing to the case that X is diagonal and remember to use the hypothesis that $XY = YX$.

(65) Let k be an algebraically closed field[1] and let X be an $n \times n$ matrix with entries in k.

(a) Show that X can be written in the form $D + N$ where D is diagonalizable, N is nilpotent, and $DN = ND$.

(b) Let λ_i be the eigenvalues of X, choose $B \geq n$ and let $g(x)$ be the polynomial in Problem 53. Show that the D you have constructed is $g(X)$.

The decomposition $X = D + N$ is known as the Jordan-Chevalley decomposition of X. It is unique, and we will likely prove so on a future problem set.

(66) This problem is a sequel to Problem 59. You may use any results from that problem without proof. We recall the key definitions: Let R be a ring. A left R-module S is called ***simple*** if $S \neq 0$ and S has no submodules other than 0 and S. A left R-module M is said to have ***finite length*** if there is a chain of submodules $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\ell = M$ such that M_{i+1}/M_i is simple for $0 \leq i < \ell$.

Let M be a module of finite length with a chain $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\ell = M$ as above.

(a) Show that M does not contain an infinite chain $V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq \cdots$ of R-submodules, nor does it contain an infinite chain $W_1 \supsetneq W_2 \supsetneq W_3 \supsetneq \cdots$ of R-submodules. Hint: Induct on ℓ, and think about $0 \to M_j \to M \to M/M_j \to 0$.

Let $\phi : M \to M$ be an R-module homormophism.

(b) Show that, for N sufficiently large, we have $\mathrm{Ker}(\phi^N) = \mathrm{Ker}(\phi^{N+1}) = \mathrm{Ker}(\phi^{N+2}) = \cdots$. Call this common kernel K. Show furthermore that ϕ takes K to K and that the restriction of ϕ to K is nilpotent.

(c) Show that, for N sufficiently large, we have $\mathrm{Im}(\phi^N) = \mathrm{Im}(\phi^{N+1}) = \mathrm{Im}(\phi^{N+2}) = \cdots$. Call this common image I. Show furthermore that ϕ takes I to I and that the restriction of ϕ to I is invertible.

(d) Show that $M = K \oplus I$.

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[1] Jordan-Chevalley decomposition can actually be defined for perfect fields, but "diagonalizable" must be replaced by a more subtle condition called "semi-simple". Here a field k is perfect if either (1) k has characteristic 0 or (2) k has characteristic p and all elements of k have p-th roots.