Problem Set 9 (Due Friday, November 22)

(67) Let k be a field, let V be an n-dimensional k-vector space and let $\alpha : V \to V$ be a diagonalizable map with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, each of multiplicity 1. Compute the eigenvalues of $\alpha \otimes \alpha$.

(68) Recall the definition of the Jordan block $J_n(\lambda)$. Compute the Jordan canonical form of $J_2(0) \otimes J_3(0)$ as a map from $\mathbb{R}^2 \otimes \mathbb{R}^3$ to itself.

(69) Let k be a field and let V and W be vector spaces with bases $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$. Let $\tau = \sum t_{ij} e_i \otimes f_j$ be an element of $V \otimes W$. Show that we can write τ in the form $v_1 \otimes w_1 + v_2 \otimes w_2 + \cdots + v_r \otimes w_r$ if and only if the matrix $[t_{ij}]$ has rank $\leq r$.

(70) Let k be a commutative ring. Let R be a unital associative k algebra and let M be a k-module. Construct a "natural" left R-module structure on $R \otimes_k M$.

(71) Let R be a commutative ring and let S be a subset of R, containing 1 and closed under multiplication. Let M be an R-module. Define $S^{-1}M$ to consist of formal symbols $s^{-1}m$ with $s \in S$ and $m \in M$, modulo the relation that $s_1^{-1}m_1 \equiv s_2^{-1}m_2$ if there is some $s_3 \in S$ such that $s_2 s_3 m_1 = s_1 s_3 m_2$.You may assume that this is an equivalence relation. We make $S^{-1}M$ into an $S^{-1}R$ module by defining $s_1^{-1}m_1 + s_2^{-1}m_2 = (s_1 s_2)^{-1}(s_2 m_1 + s_1 m_2)$ and $(s_1^{-1}r)(s_2^{-1}m) = (s_1 s_2)^{-1}(rm)$. You may assume that this is well defined or that it makes $S^{-1}M$ into an $S^{-1}R$-module.

   (a) Show that $S^{-1}M \cong S^{-1}R \otimes_R M$. Prove this isomorphism at least as R-modules, and ideally as $S^{-1}R$ modules in the sense of Problem (70).
   (b) Give an example of a triple (R, S, M) where R is an integral domain and the map $m \mapsto 1^{-1}m$ from M to $S^{-1}M$ is not injective.
   (c) Suppose that N is an R-module and M is an R-submodule of N. Show that the natural map $S^{-1}M \to S^{-1}N$ is injective.

(72) We did this problem in class, but it was sketchy enough that I think it is worthwhile to make you redo it. Let R be a PID. Let M be an R-module with $M \cong \bigoplus R/p_j^{e_j} \oplus R^{\oplus r}$ where each p_j is prime and $e_j \in \mathbb{Z}_{>0}$.

   (a) Let π be a prime element of R and let k be the field $R/\pi R$. Compute the dimension of $\pi^k M/\pi^{k+1}M$ as a k-vector space.
   (b) Suppose that $\bigoplus R/p_j^{e_j} \oplus R^{\oplus r} \cong \bigoplus R/q_j^{f_j} \oplus R^{\oplus s}$ where the q_j are prime elements and the f_j are positive integer exponents. Show that $r = s$ and that there is some permutation σ and some list of units u_j such that $q_j = u_j p_{\sigma(j)}$ and $f_j = e_{\sigma(j)}$.

(73) Let's prove that a real symmetric matrix is diagonalizable! In this problem, you may assume that the irreducible polynomials in $\mathbb{R}[x]$ are (1) the linear polynomials and (2) the quadratics $ax^2 + bx + c$ with $b^2 - 4ac < 0$.

   (a) Let X be an $n \times n$ real matrix and suppose that X is **not** diagonalizable. Prove that there is a two dimensional subspace V of $\mathbb{R}^n$ such that X takes V to itself by a matrix of either the form $\left[\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 0 & -c \\ 1 & -b \end{smallmatrix}\right]$ with $b^2 < 4c$.
   (b) Show that $\left[\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right]$ is similar to $\left[\begin{smallmatrix} 0 & -\lambda^2 \\ 1 & 2\lambda \end{smallmatrix}\right]$. Deduce that we may modify the conclusion of the previous part to say that there is a two dimensional subspace V of $\mathbb{R}^n$ such that X takes V to itself by a matrix of the form $\left[\begin{smallmatrix} 0 & -c \\ 1 & -b \end{smallmatrix}\right]$ with $b^2 \leq 4c$.
   (c) Now suppose that X is symmetric. Let $\cdot$ be the ordinary dot product on $\mathbb{R}^n$. Show that, for any v and $w \in \mathbb{R}^n$, we have $(Xv) \cdot w = v \cdot (Xw)$.
   (d) Now suppose that X is symmetric and non-diagonalizable. Let v, w be a basis of V in which X acts by the matrix $\left[\begin{smallmatrix} 0 & -c \\ 1 & -b \end{smallmatrix}\right]$ with $b^2 \leq 4c$. Show that $w \cdot w + b(v \cdot w) + c(v \cdot v) = 0$.
   (e) Deduce a contradiction.

(74) This problem is a follow up to Problems 64 and 65. You may use the results from those problems without proof. Let k be an algebraically closed field and let X be an $n \times n$ matrix with entries in k. In Problem 65, you constructed a diagonalizable matrix D and a nilpotent matrix N such that $X = D + N$, $DN = ND$ and D was $g(X)$ for a polynomial g. Now, suppose we had a second decomposition $X = D_2 + N_2$ with $D_2 N_2 = N_2 D_2$ such that D_2 is diagonalizable and N_2 is nilpotent.

   (f) Show that $D_2 X = X D_2$ and $N_2 X = X N_2$. Show that $D_2 D = D D_2$, $D_2 N = N D_2$, $N_2 D = D N_2$ and $N_2 N = N N_2$.
   (g) Show that $D - D_2$ is diagonalizable. Hint: Look at Problem 64.
   (h) Show that $N - N_2$ is nilpotent.
   (i) Show that $D - D_2 = N - N_2 = 0$.