TENSORS IN PHYSICS

"*[the torques] τ_{ij} must transform as a tensor – this is our definition of a tensor.*"
Richard Feynmann, *The Feynmann Lectures on Physics*, Volume II, Lecture 31

"*Define $V \otimes W$ to be the k-vector space generated by symbols $v \otimes w$, for $v \in V$ and $w \in W$, modulo the following relations:*"
David E Speyer, Worksheet on tensor products of vector spaces, Math 593

"*Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.*"
Johann Wolfgang von Goethe

Let k be a field and let V be a finite dimensional vector space. Let e_i and f_j be two bases of V, related by[1]

$$f_j = \sum_i a_j^i e_i.$$

(203) Let $\tau \in V \otimes V$ be given in the two bases $e_{i_1} \otimes e_{i_2}$ and $f_{j_1} \otimes f_{j_2}$ by

$$\tau = \sum_{i_1,\, i_2} T^{i_1,\, i_2} e_{i_1} \otimes e_{i_2} = \sum_{j_1,\, j_2} U^{j_1,\, j_2} f_{j_1} \otimes f_{j_2}.$$

Show that

$$T^{i_1,i_2} = \sum_{j_1,j_2} a_{j_1}^{i_1} a_{j_2}^{i_2} U^{j_1,j_2}.$$

Physicists **define** a rank two tensor as an array of numbers that transform by the formula of Problem (203) and speak similarly of higher rank tensors.

Let $V^\vee$ be the dual vector space to V. Let e^i and f^j be the corresponding dual bases of $V^\vee$, meaning that[2]

$$\langle e^p, e_q \rangle = \begin{cases} 1 & p = q \\ 0 & p \neq q \end{cases} \text{ and likewise for the } f\text{'s.}$$

(204) Show that

$$f^j = \sum_i (a^{-1})_i^j e^i$$

where a^{-1} is the inverse matrix to a.

(205) Let $\tau \in V \otimes V^\vee$ be given in the two bases $e_{i_1} \otimes e_{i_2}$ and $f_{j_1} \otimes f_{j_2}$ by

$$\tau = \sum_{i_1,\, i_2} T_{i_2}^{i_1} e_{i_1} \otimes e^{i_2} = \sum_{j_1,\, j_2} U_{j_2}^{j_1} f_{j_1} \otimes f^{j_2}.$$

Show that

$$T_{i_2}^{i_1} = \sum_{j_1,j_2} a_{j_1}^{i_1} (a^{-1})_{j_2}^{i_2} U_{j_2}^{j_1}.$$

(206) How does the formula in Problem 205 simplify if you assume that the matrix a_i^j is orthogonal?

Textbooks in differential geometry, general relativity or quantum field theory often need both vector spaces and their duals, and use upper and lower indices as in the examples above. This issue rarely comes up in non-relativistic physics, as most changes of coordinates occurring there are orthogonal.

Meanwhile, people working in quantum mechanics encounter tensors in a very different way. Turn over the worksheet to read about it!

[1] Note that, in this equation, the indices of summation appear exactly once in the top and once in the bottom. If you set up your notation right, you can make sure this is always true, which is a great check against errors. Physicists often take this a step further and consider any repeated index of this sort to carry an implicit summation, so they would write $f_j = a_j^i e_i$. This is called the ***Einstein summation convention***; we will not use it.

[2] A physicist would write more briefly $e^p \cdot e_q = \delta_q^p$. This notation is called the ***Kronecker delta.***

Quantum computers are built out of qubits. Rather than a qubit being 0 or 1, its internal state is represented by a vector $\left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right] \in \mathbb{C}^2$ obeying $|z_0|^2 + |z_1|^2 = 1$. If we measure it, either get the output 0 with probability $|z_0|^2$ or get the output 1 probability $|z_1|^2$. After measurement, it will either be in state $|0\rangle$ or $|1\rangle$, matching our measurement.

Rather than writing $\left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right]$, a physicist would write $z_0|0\rangle + z_1|1\rangle$. If we have n qubits, their state is encoded by a vector in $(\mathbb{C}^2)^{\otimes n}$. Physicists abbreviate $|0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle$ to $|0100\rangle$. Manipulations performed on the k-th qubit act by linear operators on the k-th tensor factor. Again, if our qubits are in state $\sum_{i_1,i_2,\dots,i_n \in \{0,1\}^n} z_{i_1 i_2 \cdots i_n} |i_1 i_2 \cdots i_n\rangle$, we will get the output $(i_1, i_2, \dots, i_n)$ with probability $|z_{i_1 i_2 \cdots i_n}|^2$.

(207) Alice and Bob are each holding one qubit, and the state of their qubits is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. If Alice modifies her qubit by $\left[\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right]$ and Bob modifies his qubit by $\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$, compute the resulting state of their qubits.

(208) After performing the above actions, Alice and Bob measure the state of their qubits. Show that the probability that they obtain the same measurement is $\cos^2(\alpha - \beta)$.

This last result seems innocuous and it is experimentally correct – even when Alice and Bob are kilometers are apart. But it is the basis of a result known as Bell's theorem about which a distinguished Princeton physicist was quoted to say "Anybody who's not bothered by Bell's theorem has to have rocks in his head."[1] Let's try to see why:

(209) Let's begin with something not at all mystifying. Alice and Bob pick up their quibits, in state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and walk a mile apart. Each of them then chooses to rotate their qubit by either 0° or 90°, and then performs a measurement. The probability that their measurements agree, in terms of the rotation angle, is

$$\begin{array}{c|cc} & 0^\circ & 90^\circ \\ \hline 0^\circ & 1 & 0 \\ 90^\circ & 0 & 1 \end{array}.$$

This should **not** seem strange; describe a very boring way this could happen.

(210) Now Alice and Bob choose rotation angles of 0°, 60° or 120°. The new probabilities are

$$\begin{array}{c|ccc} & 0^\circ & 60^\circ & 120^\circ \\ \hline 0^\circ & 1 & \frac{1}{4} & \frac{1}{4} \\ 60^\circ & \frac{1}{4} & 1 & \frac{1}{4} \\ 120^\circ & \frac{1}{4} & \frac{1}{4} & 1 \end{array}.$$

Explain why this is incredibly disturbing.

[1] N. David Mermin, "Is the Moon there when nobody looks? Reality and the quantum theory," *Physics Today*, April 1985